NYSMATYC MATH LEAGUE COMPETITION
Fall 2007
ANSWER SHEET

Directions: You have one full hour to take this test. Scrap paper is allowed. The use of a calculator is permitted, but not stored programs on the calculator. Moreover, books, tables, and computers are not permitted. You are not expected to answer all problems. Yet, do not waste too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and nothing is deducted for a blank. There is no partial credit. Note: Some answers are asked to be in exact form. Exact answers must be expressed as fraction or a radical or in terms of π, depending on context. If not specified then answers may be given in fractional form, radical form, or as an approximation to the third decimal place as appropriate. For example, √7 or 2.646.

Name __KEY__ College __________________________
Home Address ____
(include, city, state, zip code)
Name of teacher in whose class you are enrolled __________________________

1. C	11. 1/60
2. 9	12. 6
3. B	13. $10\sqrt{601}$
4. D	14. 2
5. Yellow	15. Zero
6. C	16. 96
7. a = 6 & b = 8	17. C
8. C	18. 47/7
9. 8	19. 48π un.^{2}
10. 120 min.	20. 1

Correct ____________________ X 4 = ______________

Incorrect ____________________ X –1 = ______________

Total = ______________

Math Contest Fall 2007

1. A rectangular solid slice of cheese is 3 inches by 3 inches and $\frac{1}{8}$ inch thick. If the slice is cut by 9 straight cuts forming rectangular pieces of random width perpendicular to the 3 x 3 face, by what percent is the surface area increased to the nearest whole number percent?

   A. 11% B. 30% C. 35% D. 45%

2. If $a \circ b * c = \frac{a+b}{b-c} \times c$, then find the value of $5 \circ 7 * 3$.

3. An $11 \times 11 \times 11$ cube is formed by gluing together 11^3 unit cubes. What is the greatest number of unit cubes that can be seen from a single point?

   A. 332 B. 331 C. 330 D. 329

4. Which of the following equations have the same graph?

   I. $y = x - 3$ II. $y = \frac{x^2 - 9}{x + 3}$ III. $(x + 3)y = x^2 - 9$

   A. I and II only B. I and III only C. I, II, and III
   D. None. All equations have different graphs.

5. A bag contains marbles that are red, green, blue, or yellow. They are identical in size, have different numbers of each, and a single marble is drawn. Three people are asked to guess the color of the marble that is drawn. Otto says, "It is red or green." Oriole says, "Two marbles were drawn." Clovis says, "It is not yellow or it is blue." If all three statements are not true, then what is the color of the marble?

6. From time $t = 0$ to time $t = 1$, the population of Normalsburg increased by $i\%$. From time $t = 1$ to time $t = 2$ the population on Normalsburg increased by $j\%$. Therefore, from time $t = 0$ to time $t = 2$, the population increased by

A. $(i + j)\%$ B. $ij\%$ C. $(i + j + \frac{ij}{100})\%$ D. $(i + j + \frac{i+j}{100})\%$

7. What are the smallest values of a and b such that $21_b = 25_a$ where 21_b denotes two-one base b and 25_a denotes two-five base a?

8. How many different numbers are in the solution set of

$$x^2 - 9 = |x + 3|?$$

A. 0 B. 1 C. 2 D. 4

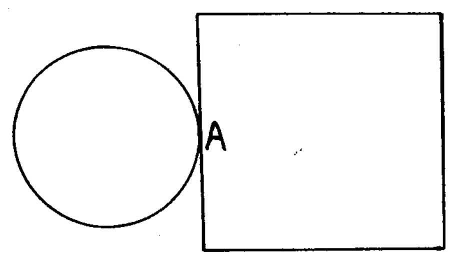
9. What is the units digit of the number 1492^{2007}?

10. In a twelve hour time period of time, for how many minutes is the sum of the digits on an electronic digital clock greater than 15?

11. A five digit number is chosen at random from a table of all five digit numbers. What is the probability that the number is evenly divisible by 1, 2, 3, 4, and 5?

12. Let $f(x) = 2^x$ and let $g^{-1}(x) = \frac{\log_4 x}{3}$. Find $f^1(g(1))$.

13. At the point A the circle is tangent to the square at the midpoint of a side of the square as shown in the diagram. The radius of the circle is 70 meters and a side of the square is 100 meters. Tom and Susan start out from point A at the same time. Tom jogs clockwise around the square at $\frac{50}{21}$ meters per second while Susan walks counter-clockwise around the circle at $\frac{2\pi}{3}$ meters per second. Exactly how far apart are they $12\frac{1}{4}$ minutes after they started?



14. A certain number when divided by 91 yields a remainder of 52. If we add 104 to the certain number, what is the reminder when the new number is divided by 7?

15. At how many points do the graphs of the two functions defined below intersect?

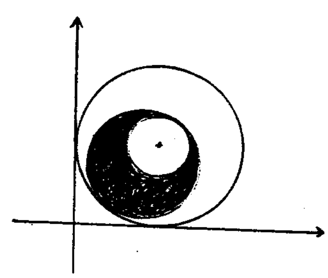
$$y = \begin{cases} 10^{2007} - x^2 & \text{if } x < 2 \\ \sqrt{x} + 10^{2007} & \text{if } x \geq 2 \end{cases} \qquad \text{and } y = 10^{-x} + 10^{2007}$$

16. In how many of the following numbers is 144 a factor?
$1!$, $2! + 3!$, $3! + 4! + 5!$, $4! + 5! + 6! + 7!$, $5! + 6! + 7! + 8! + 9!$, …, $100! + 101! + 102! + \ldots + 199!$?

17. p, r, and t are statements and the statement $(p \wedge \sim r) \rightarrow [\sim r \leftrightarrow (\sim p \vee t)]$ is false, then how many of p, r, and t must be false?
A. None of them B. Exactly one of them C. Exactly two of them
D. All three of them

18. Data set A has a mean of 6. Data set B has a mean of 3. Data set C has a mean of 8. When data sets A and B are put together, the mean of the resulting set is 5. When data sets B and C are put together, the mean is 7. What is the mean when all three data sets are combined?

19. The 3 circles shown have their centers on the line $y = x$. The largest and smallest circles have the same center. The middle sized circle is tangent to the other two circles as shown. The largest circle is tangent to both axes. The smallest circle goes through the center of the middle sized circle. The radius of the largest circle is 12. What is the area of the shaded region?



20. What is the exact value of the expression: $\dfrac{2^{5000}3^{4000}4^{7000}6^{8000}}{8^{9000}9^{6000}}$?

Mathematics Contest Solutions
Fall 2007

1. Since the cheese slice is 3 in. X 3 in. X $\frac{1}{8}$ in., the original surface area is the area of top and bottom plus the 4 times the area of one side that is 3 in. X $\frac{1}{8}$ in. Thus, the original surface area, S = $2(3\cdot3) + 4(3)(\frac{1}{8}) = 19\frac{1}{2}$ in.^{2}. Each slice produces two new edges that are the same area as a side or $\frac{3}{8}$ in.^{2} Since there are 9 slices there are 18 new edges of total area $18(\frac{3}{8}) = \frac{27}{4}$ in.^{2}. The percent increase is then $\frac{\frac{27}{4}}{\frac{39}{2}}(100) =$ $\frac{9}{26}(100)$ 35%. Thus, the answer is C. **Ans. C**

2. Given $a \circ b * c = \frac{a+b}{b-c} \times c$, then $5 \circ 7 * 3 = \frac{5+7}{7-3}(3) = 9$. **Ans. 9**

3. If we look at the cube from above the top center corner we can two sides and the top. From any other angle we can see no more than three faces. On the first face we will see 11 rows and columns of cubes for a total of 11(11) = 121 cubes. If we move to an adjacent side we will 11 cubes in one direction but on the edge formed by the two sides meeting we have already counted those 11 cubes. Thus, there are 11(10) = 110 new cubes. On the remaining side there are two edges formed by the each of the first two sides meeting the third side. We have to subtract off the cubes that were previously counted. This leaves 10(10) = 100 new cubes. Altogether, we have 121 + 110 + 100 = 331 cubes visible. **Ans. B**

4. For I, the graph of $y = x - 3$ is an oblique line that is defined every where. For II, $y = \frac{x^2 - 9}{x + 3}$, the graph is an oblique line that is undefined at $x = -3$. So the graph has a hole at $x = -3$. For III, $(x + 3)y = x^2 - 9$, the equation is defined everywhere. Thus, we have the oblique line of I. However, at $x = -3$, the equation becomes $0y = 0$ for which y can be any real number. This give the vertical line $x = -3$ as well. Hence, each graph is different. **Ans. D**

5. We are told that all three statements are false. Since a single draw was made, we know Oriole's statement is false. Since Otto said "It is red or green", the statement will be false only if both parts are false. Thus, we know that the draw was not red nor was it green. Similarly, Clovis said, " It is not yellow or it is blue." Since both parts must be false, we know the draw was not blue and for "not yellow" to be false, the draw must have been yellow. **Ans. Yellow**

6. Let the original population of Normalsburg be P. The increase from time 0 to time 1 is P(.01i) = .01iP. Thus, the new population is P + .01iP = P(1 + .01i). From time 1 to time 2, the increase is P(1 + .01i)(.01j). Again, the new population would be P(1 + .01i) + P(1 + .01i)(.01j) = P(1 + .01i)(1 + .01j). The percent increase is the given by

$\frac{P(1+.01i)(1+.01j)-P}{P} \cdot 100 = ((1+.01i)(1+.01j)-1)(100) = (1+.01i+.01j+.0001ij-1)(100) = (.01i+.01j+.0001ij)(100) = (i+j+\frac{ij}{100})\%$. **Ans. C**

7. $21_b = 2b + 1$ while $25_a = 2a + 5$. Since these values are equal, we have $2b + 1 = 2a + 5$ or $b = a + 2$. The base in any number scheme has be greater than the largest digit so a must be at least 6. Since we are looking for the smallest values, we let $a = 6$ and then $b = 8$. **Ans. a = 6 , b = 8**
8. If $x \geq -3$, $|x + 3| = x + 3$. If $x < -3$, $|x + 3| = -(x + 3)$. For $x \geq -3$, we have $x^2 - 9 = x + 3$.
   Then, $(x + 3)(x - 3) = x + 3$, or, $(x + 3)(x - 3) - (x + 3) = 0$. Factoring we have $(x + 3)(x - 3 - 1) = 0$ or, $(x + 3)(x - 4) = 0$ which has solutions -3 and 4. For $x < -3$, we have $x^2 - 9 = -(x + 3)$ or, $(x + 3)(x - 3) + (x + 3) = 0$. Factoring, we have $(x + 3)(x - 3 + 1) = 0$ or, $(x + 3)(x - 2) = 0$, which has solutions -3 and 2. But since $x < -3$, we have to eliminate these two solutions. Thus, there are three unique solutions. **Ans. C**
9. In 1492^{2007} note that the units digit is entirely determined by the 2 in the units position. If we look at some powers of two we can see a pattern. $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, … We see that the units digit follows the pattern, 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, … Every fourth number has the same digit. Thus, to find the units digit in 1492^{2007}, we divide 2007 by 4 and look at the remainder. Since the remainder is three the units digit is the same as $2^3 = 8$. **Ans. 8**
10. Assuming that the clock displays only hours and minutes, we can consider the possibilities. If we start with the times in the 12 o'clock hour, the first two digits add to 3 so the remaining digits need to add to at least 13. The right most digit varies from 0 to 9 while the digit to its right will display the digits from 0 to 5. In order to have a sum of at least 13, we could have 49, 59, or 58. At the 11 o'clock hour, the first two digits add to 2, so we need at least 14 for the next two. In this case only11: 59 works. At the 10 o'clock hour the first two digits add to 1 so the remaining digits would need to add to at least 15 which is not possible. Thus, there are four minutes from 10 to twelve that satisfy the criterion. For the hours between 1 and 9 inclusive during the one o'clock hour the first digit is one and the remaining two digits would have to add to at least 15 which again is not possible. For the two o'clock hour, the remaining digits would have to add to at least 14 which is only 59, one minute. Similarly, at three we would need at least 13 which is 39, 58, and 59, three minutes. At four we would need at least 12 which adds 39, 48, and 57 to the previous for a total of 6 minutes. For five we would need at least 11, which adds 29, 38, 47, and 56 to the previous for a total of 10 minutes. At 6 we would need at least 10, which adds 19, 28, 37, 46, and 55 to the collection for a total of 15 minutes. At 7 we would need at least 9 which adds 18, 27, 36, 45, 54, and 09 for a total of 21 minutes. At 8 we would need at least 8 more which extends the list to 08, 17, 26, 35, 44, and 53 for a total of 27 minutes. Lastly, at the 9 o'clock hour we would need to have at least 7 which adds 07, 16, 25, 34, 43, and 52 for a total of 33 minutes. Upon adding all the possibilities we get a total of 120 minutes. **Ans. 120 min.**

11. In order for a number to be divisible by those numbers listed it must have all the

numbers as factors. Thus, the number has to be a multiple of $3 \cdot 4 \cdot 5 = 60$. The smallest multiple of 60 that is greater than 10000 is 10020 and the largest number that is less than 99999 is 99960. These two number and all the numbers in between form an arithmetic sequence of the form $10020 + 60(n - 1)$. In order to find the number of terms in this sequence we can solve the following equation: $10020 + 60(n - 1) = 99960$. The solution is that $n = 150$. Hence, the probability of selecting one of these numbers from the table is $\frac{150}{90000} = \frac{1}{60}$. **Ans.** $\mathbf{\frac{1}{60}}$

12. If $f(x) = 2^x$, then $f^{-1}(x) = \log_2 x$. If $g^{-1}(x) = (\log_4 x)/3$, then we can find $g(x)$ AS FOLLOWS. Let $y = (\log_4 x)/3$. I f we interchange x and y, then we have $x = (\log_4 y)/3$. If we now solve for y, we have $\log 4y = 3x$ or , equivalently, $y = 4^{3x}$. Thus, $g(x) = 4^{3x}$. Then, $f^{-1}(g(1)) = f^{-1}(64) = \log_2 64 = 6$. **Ans. 6**

13. Each of the walking rates is given in m/s. 12 ¼ minutes is 735 sec. In that time Tom jogs $\frac{50m}{21s}(735\text{ s}) = 1750$ m. Since the perimeter of the square is $4(100) = 400$m, Tom will go 150 meters past his starting point. He will be on the upper right hand corner of the square. For the same time period Susan walks $\frac{2\pi m}{3s}(735\text{ s}) = 490\pi$ m. Since the circumference of the circle is $2\pi(70) = 140\pi$ m, Susan will make $\frac{490\pi m}{140\pi m} = \frac{7}{2}$ trips around the circle. Hence, she will on the far left side of the circle opposite the point A. The horizontal distance from her to Tom is $140 + 100 = 240$ m. The vertical distance between her and Tom is 50 m. Thus, the distance, d, between the two will be $d = \sqrt{240(240)+50(50)} = \sqrt{60100} = 10\sqrt{601}$ m.

Ans. $\mathbf{10\sqrt{601}}$ **m**

14. If we let x = the certain number, $x = 91q + 52$ for some whole number, q. Upon adding 104 to x we have $x + 104 = 91q + 52 + 104 = 91q + 156$. When we divide this number by 7 we have $\frac{91q}{7} + \frac{156}{7} = 13q + 22 + \frac{2}{7}$. Thus, the remainder is 2.

Ans. 2

15. If we consider first the graph of $y = 10^{-x} + 10^{2007}$, it is the graph of a decaying exponential function whose y-intercept is $(0, 1 + 10^{2007})$ and it has a horizontal asymptote to the right of the y axis of $y = 10^{2007}$. For the other function for $x < 2$, the function is $y = 10^{2007} - x^2$ which is part of a parabola that opens down and whose maximum is at $(0, 10^{2007})$. Hence, it does not get to the graph of the first function and there is no intersection point. For $x \geq 2$, we have $y = \sqrt{x} + 10^{2007}$. The graph of this function is a parabola that opens to the right and has its minimum value at $(\sqrt{2}, \sqrt{2} + 10^{2007})$. Since $\sqrt{2} > 1$, this graph starts above the first one considered and increases to

the right while the first is decreasing. Hence, there are no points of intersection.

Ans. Zero

16. Firstly, if we look at $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 6(2)(4\cdot3)5 = 12^2(5) = 144(5)$. Hence, 144 is a factor of 6!. Further, it is a factor of any factorial greater than 6! As 7! = 7(6!), etc. Thus, 144 is a factor of every term starting from 6! +7! +8! +9! + 10! + 11!. There are 95 such terms. It remains to see if 144 is a factor of any of the other terms. Clearly we can eliminate the first two terms. 3! + 4! + 5! = 6 +24 +120 = 150. So 144 is not a factor of the third term. 4! + 5! + 6! + 7! = 4!(1 + 5 + 30 + 210) = 24(246) = 2(12)(2)(3)(41) = 144(41). So 144 is a factor of that term. However, 144 is not a factor of 5! + 6! + 7! + 8! + 9!. Hence, there are 96 terms of which 144 is a factor.

Ans. 96

17. Since the given statement is a conditional statement it is false only when the premise is true and the conclusion is false. For the premise p ∶ r to be true we need to have both parts true. Hence, p is true and r is false. The conclusion is a biconditional which must be false. Since ∶ r is true, we need to have ∶ p t to be false. Since ∶ p is false as p is true, we need t to be false. Thus, r and t must be false statements.

Ans. C

18. Since data set A has mean 6 and if we let a be the number of data values in the set, then the sum of all scores in data set A is 6a. Similarly, we can express the sum of all scores in data set B as 3b and the sum of all scores in data set C as 8c. If we combine data sets A and B we will a + b data values and their sum will be 6a + 3b. then the mean is $\frac{6a+3b}{a+b} = 5$. Then 6a + 3b = 5a + 5b. On solving this equation we find that a = 2b. If we combine data sets B and C we will have b + c data values whose sum is 3b + 8c. Then their mean will be $\frac{3b+8c}{b+c} = 7$. On solving this equation we have 3b + 8c = 7b + 7c or c = 4b. When we combine all three data sets, we will get $\frac{6a+3b+8c}{a+b+c}$. Substituting our previous results, we get $\frac{6(2b)+3b+8(4b)}{2b+b+4b} = \frac{12b+3b+32b}{7b} = \frac{47b}{7b} = \frac{47}{7}$.

Ans. $\frac{47}{7}$

19. Since the smallest and largest circles have the same center as shown, we can see that if let r be the radius of the smallest circle, then the radius of the middle circle must be 2r. Also, 2r + r = 12, the radius of the largest circle. From this last equation we conclude that the smallest circle has radius 4 units and the middle circle has radius 8 units. The shaded area is the difference of the areas of the middle circle and the smallest circle. Hence, $8^2\pi - 4^2\pi = 64\pi - 16\pi = 48\pi$ units^2.

Ans. 48π un.^{2}

20. To find the exact value we need to express each term with bases of two and/or three. $4^{7000} = (2^2)^{7000} = 2^{14000}$. $6^{8000} = (2\cdot3)^{8000} = 2^{8000}3^{8000}$. Upon subbing these values into the numerator we have for the numerator $2^{5000+14000+8000}3^{4000+8000} = 2^{27000}3^{12000}$. In the denominator, $8^{9000} = (2^3)^{9000} = 2^{27000}$ and $9^{6000} = (3^2)^{6000} = 3^{12000}$. When we replace the

original fraction with these values we have $(2^{27000} \cdot 3^{12000})/(2^{27000} \cdot 3^{12000}) = 1$.

Ans. 1