NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2010

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

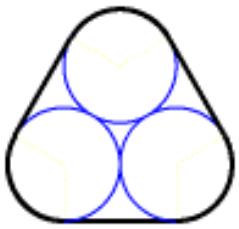
1. If $f(x,n)=\sum_{k=0}^{n} x^k = x^0 + x^1 + x^2 + \ldots + x^n$, what is $f(i,2010)$ where $i=\sqrt{-1}$? Express the answer in simplest $a+bi$ form.

2. What value of x satisfies: $\sqrt{2010x+1} - \sqrt[4]{2010x+1} = 2$?

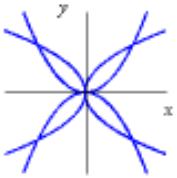
3. $\sqrt{7+\sqrt{48}} + \sqrt{7-\sqrt{48}}$ equals
   a) $\sqrt{14}$ b) $\sqrt{15}$ c) $\sqrt{16}$ d) $\sqrt{17}$

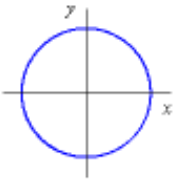
4. Rich has an assortment of coins but is unable to give exact change for a dollar, half-dollar, quarter, dime, or nickel. From the set of coins {pennies, nickels, dimes, quarters, half-dollars}, what is the largest amount he can have?
   a) \$0.94 b) \$0.99 c) \$1.19 d) none of those answers

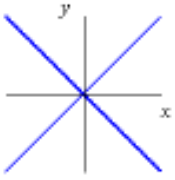
5. Three large pipes are transported on a flatbed truck by stacking the third one on top of the other two and using a tight metal band around all three to hold them together (an end-view is shown). If the outside diameter of each pipe is 2 feet, then what is the length of the metal band, assuming it does not overlap itself?

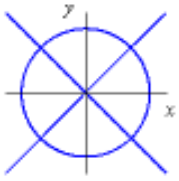


6. Which of the following is the graph of $x^4 - y^4 = x^2 - y^2$?

a) 

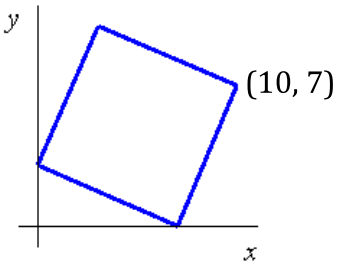
b) 

c) 

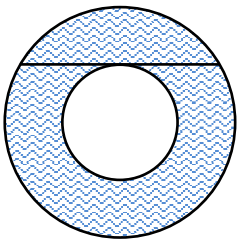
d) 

7. If a and b are positive integers such that $a^2 - b^4 = 345$, then what is the value of $a + b$?

8. A square is constructed so one corner lies on the x-axis, one on the y-axis, and one at (10, 7), as shown. What is the area of the square?



9. The diagram shows two circles with a common center, and a chord drawn in the larger circle so it is tangent to the smaller circle. The radius of the larger circle is R, the radius of the smaller circle is r, while the chord has length 2. What is the area of shaded region (i.e. the area between the two circles)?



a) π b) 2π c) $2\pi(R-r)$ d) $\pi(R-r)^2$

10. At a fundraiser 2010 raffle tickets were sold, numbered 1 through 2010. Three prizes were awarded by randomly selecting three of the ticket stubs. What is the probability that the numbers on the three stubs were selected in increasing order?

11. The graph of $y = \dfrac{x}{100\pi}$ intersects the graph of $y = \sin(x)$ at how many points?

12. There is only one number between 2 and 2^{100} whose square root, third root, fourth root, fifth root, and sixth root is a whole number. What is the number?

13. The exact value of $\log_{2011}(2010) - \dfrac{1}{\log_{2010}(2011)}$ is

a) -1 b) 0 c) 1 d) $\dfrac{1}{2010+2011}$

14. Ray can do a job in 9 hours. Ray and his helper Tim start working together, after 4 hours Tim leaves and Ray finishes the job alone in 2 hours. How many hours would it take Tim to do the entire job alone?

15. A coin is so unbalanced that the probability of getting two heads on two tosses is the same as getting tails on one toss. What is the probability of getting heads on a single toss?

16. I wear a clean shirt every day. If I drop off my laundry and pick up the previous week's load every Monday evening, then what is the minimum number of shirts I must own to keep me going?
    a) 13 b) 14 c) 15 d) 16

17. Ida and Sophia were in a bicycle race and started at the same time. Ida was travelling at a constant speed of 42 km/hr. When Sophia completed half the race, Ida was 3.5 km ahead. Ida completed the race at 11:52 AM. If Sophia also maintained a constant speed, what time did she complete the race?

18. A handball tournament consists of 2010 participants. The format of the event is single-elimination (i.e. a player is out of the tournament after the first loss). How many matches must be played in order to determine a winner?
    a) 1005 b) 2009 c) 2010 d) 2011

19. The citizens of Taxation Land pay twice as much income tax (percentage-wise) as their hourly wage. For example, someone who earns $15 per hour would pay 30% income tax. What hourly wage maximizes after taxes income?

20. During a recent mathematics lecture, someone in the audience created a disturbance with loud snoring. In an effort to determine the culprit, the five most likely suspects were interrogated and each offered two statements.

Artie:	It wasn't Edward. It was Barbara.
Barbara:	It wasn't Carmine. It wasn't Edward.
Carmine:	It wasn't Artie. It was Edward.
Darci:	It was Carmine. It was Barbara.
Edward:	It wasn't Artie. It was Darci.

    If each suspect told exactly one lie, then who committed the heinous act?

NYS∫∞MATYC 1967

NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2010 ~ Solutions

1. $f(i,2010)=\sum_{k=0}^{2010} i^k = i^0+i^1+i^2+i^3+i^4+\ldots+i^{2007}+i^{2008}+i^{2008}+i^{2009}+i^{2010}$

$=(\underbrace{1+i-1-i}_{k=0,1,2,3})+(\underbrace{1+i-1-i}_{k=4,5,6,7})+\ldots+(\underbrace{1+i-1-i}_{k=2004,2005,2006,2007})+1+i-1=\underbrace{0+0+\ldots+0}_{502\text{ zeros}}+1+i-1=i$

Or use the formula for a geometric series, $\sum_{k=0}^{n} r^k=\frac{r^{n+1}-1}{r-1}$ with $r=i$ and $n=2010$. Answer: i

2. If we let $y=\sqrt[4]{2010x+1}$, then $\sqrt{2010x+1}=y^2$. Now the equation takes on a simpler form: $y^2-y=2$. Solving for y: $y^2-y-2=0\Rightarrow(y+1)(y-2)=0\Rightarrow y=-1$ or $y=2$. However, y must be positive. Thus, $y=2\Rightarrow\sqrt{2010x+1}=2^2\Rightarrow 2010x+1=4^2\Rightarrow 2010x=15$.

Therefore, $x=\frac{15}{2010}=\frac{1}{134}$. Answer: $\frac{15}{2010}$ or $\frac{1}{134}$

3. Let $x=\sqrt{7+\sqrt{48}}+\sqrt{7-\sqrt{48}}$, which is certainly greater than zero. Squaring gives:

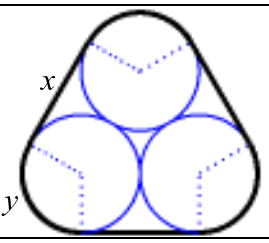
$x^2=\left(\sqrt{7+\sqrt{48}}+\sqrt{7-\sqrt{48}}\right)^2\Rightarrow x^2=\left(\sqrt{7+\sqrt{48}}\right)^2+2\left(\sqrt{7+\sqrt{48}}\right)\left(\sqrt{7-\sqrt{48}}\right)+\left(\sqrt{7-\sqrt{48}}\right)^2$

$=7+\sqrt{48}+2\sqrt{49-48}+7-\sqrt{48}=14+2\sqrt{1}=16$ Taking the positive root, $x=\sqrt{16}$. Answer: c

4. Rich would have 4 pennies, 4 dimes, 1 quarter and 1 half-dollar. Answer: \$1.19

5. Let x be the length of the straight segment between two circles, and let y be the length of the arc from one straight segment to the other.
x = 2·radius = 2·1 = 2 ft, and y = (⅓)·circumference = (⅓)·2π = (⅔)π ft
Thus, the total length = $3x + 3y$ = 3·2 + 3·(⅔)π = 6 + 2π ft.

Answer: 6 + 2π ft



6. $x^4-y^4=x^2-y^2\Rightarrow\left(x^4-y^4\right)-\left(x^2-y^2\right)=0\Rightarrow\left(x^2-y^2\right)\left(x^2+y^2\right)-\left(x^2-y^2\right)=0$

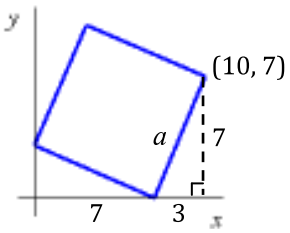
$\Rightarrow\left(x^2-y^2\right)\left[\left(x^2+y^2\right)-1\right]=0\Rightarrow x^2-y^2=0$ or $x^2+y^2-1=0$

$\Rightarrow y^2=x^2\Rightarrow y=\pm x$ or $x^2+y^2=1$, which gives two lines through the origin with slopes of -1 and 1, and a circle of radius 1 centered at the origin. Answer: d

7. $a^2-b^4=345\Rightarrow\left(a-b^2\right)\left(a+b^2\right)=3\cdot5\cdot23$, grouping the three factors of 345 in pairs give $(3\cdot5)\cdot23$, $3\cdot(5\cdot23)$, and $5\cdot(3\cdot23)\Rightarrow15\cdot23$, $3\cdot115$, and $5\cdot69$. Thus, either

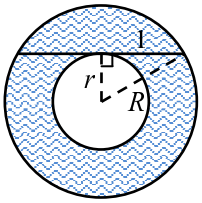
❶ $a-b^2=15$ and $a+b^2=23$, ❷ $a-b^2=3$ and $a+b^2=115$, or ❸ $a-b^2=5$ and $a+b^2=69$

Only equations ❶ yield an integer for both a and b, with $a=19$ and $b=2$. Answer: $a+b=21$

8. 

Letting a be the length of one side of the square, we get:
$a^2 = 3^2 + 7^2 \Rightarrow a^2 = 9 + 49 = 58 =$ Area of the Square

Answer: 58

9. 

The area of the shaded region is: $\pi R^2 - \pi r^2 = \pi\left(R^2 - r^2\right)$.
From the diagram, we get the relation: $R^2 = r^2 + 1 \Rightarrow R^2 - r^2 = 1$.
Thus, the area of the shaded region is $\pi \cdot 1 = \pi$.

Answer: a

10. Three tickets can be selected in 3! ways, or 6 ways. Only 1 of the 6 permutations would have the tickets in increasing order.

Answer: ⅙

11. Since the maximum value of $\sin(x)$ is 1, the line will not intersect the sine curve for $x > 100\pi$. Thus, from $x = 0$ to $x = 100\pi$, there will be 2 points of intersection per period. The period of $y = \sin(x)$ is 2π, which means there are 50 periods from $x = 0$ to $x = 100\pi$. Hence, there will be 50·2=100 points of intersection from $x = 0$ to $x = 100\pi$. Similarly from $x = -100\pi$ to $x = 0$, but that would count the origin again. Therefore, there are $100 + 100 - 1$ points of intersection.

Answer: 199

12. If there is only one such number between 2 and 2^{100}, then it must be the smallest. Hence, it must be 2^n, where n is the least common multiple of 2, 3, 4, 5 and 6 (so the 2nd, 3rd, 4th, 5th and 6th roots are whole numbers). Thus, $n = 3 \cdot 4 \cdot 5 = 60$ and the number is 2^{60}.
Note: The next such number is $2^{2 \cdot 60} = 2^{120} > 2^{100}$.

Answer: 2^{60}

13. $\log_{2011}(2010) - \dfrac{1}{\log_{2010}(2011)} = \dfrac{\log(2010)}{\log(2011)} - \dfrac{1}{\left(\dfrac{\log(2011)}{\log(2010)}\right)} = \dfrac{\log(2010)}{\log(2011)} - \dfrac{\log(2010)}{\log(2011)} = 0$

Answer: b

14. Ray works at the rate of $\frac{1}{9}$ of a job/hr. Let r be the rate at which Tim works. Together, they work at $\left(\frac{1}{9} + r\right)$ job/hr, and since Work = Rate · Time, after 4 hours they complete $4\left(\frac{1}{9} + r\right)$ of the job. However, it takes Ray 2 more hours to finish. Thus, there was only $2\left(\frac{1}{9}\right) = \frac{2}{9}$ of the job remaining. Therefore, together they must have completed $\frac{7}{9}$ of the job. From this we get: $4\left(\frac{1}{9} + r\right) = \frac{7}{9}$. Solving for r, gives $r = \frac{1}{12}$ job/hr. Hence, working alone, Tim would need 12 hours.

Answer: 12 hours

15. Let p = the probability of obtaining *heads up* on one toss of the coin.
Then $1 - p$ = the probability of obtaining *tails up* on one toss of the coin.
The probability of getting two heads on two tosses is p^2, which is equal to $1 - p$.
Thus, $p^2 = 1 - p \Rightarrow p^2 + p - 1 = 0 \Rightarrow p = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2}$.
However, only $\dfrac{-1 + \sqrt{5}}{2}$ is positive (and between 0 and 1).

Answer: $\dfrac{\sqrt{5} - 1}{2}$

16.	I must pick up 7 shirts to hold me over until the following Monday. Hence, I must drop off 7 shirts each Monday. Counting the shirt I wear on Monday, the required total is 7+7+1=15. Note: I cannot get by with only 14 shirts, as I would not have a clean shirt to wear the following Monday. Answer: c
17.	When Sophia completed half the race, Ida was 3.5 km ahead. Thus, by the time Sophia completed the entire race, Ida would have been 7 km ahead (if she continued at the same rate). Hence, Sophia completed the race at 11:52 AM plus the time it would have taken Ida to go an additional 7 km. At 42 km/hr , it would have taken Ida $(7 \div 42)$ hour = ⅙ hour = 10 minutes. Therefore, Sophia completed the race at 12:02 PM. Answer: 12:02 PM
18.	Since each match eliminates one competitor, and 2009 competitors must be eliminated so that only one person (the winner) remains, 2009 matches must be played. Answer: b
19.	Let w represent the hourly wage. Then the tax rate is $(2w)\%$ and the tax on w will be $\frac{2w}{100} \cdot w = \frac{w^2}{50}$. Thus, the after taxes income (per hour) is $w - \frac{w^2}{50}$. This quadratic is maximized along the axis of symmetry, i.e. for $w = \frac{-b}{2a} = \frac{(-1)}{2\left(-\frac{1}{50}\right)} = 25.$ Answer: $25 per hour
20.	Look at each clue, knowing *exactly* one of each person's statements is true. Artie's: If *it was Barbara* is true, then we know the other statement is false, therefore it was Edward. This is a contradiction. Hence we now know it wasn't Barbara, nor Edward (as *it wasn't Edward* must be the true statement). Looking at Carmine's statements, we can similarly determine that it wasn't Artie. Since we know it wasn't Barbara, Darci's statements tell us *it was Carmine* is true. This also checks against the other clues. Answer: Carmine

NYS ∞ MATYC 1967