New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2011

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

1. The Ackermann function, named after the German mathematician Wilhelm Ackermann, is defined as
$$A(m,n)=\begin{cases} n+1, & \text{if } m=0 \\ A(m-1,1), & \text{if } m>0 \text{ and } n=0. \\ A(m-1, A(m,n-1)), & \text{if } m>0 \text{ and } n>0 \end{cases}$$
What is the numerical value of $A(1,2)$?

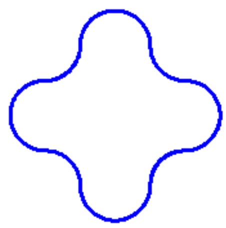
2. For what value of *k* will the parabola given by $f(x)=3x^2+kx+1$ be tangent to the *positive* *x*-axis?

3. The system of equations $x^2+y^2-2x-2y+1=0$ and $x^2+y^2-8x-10y+25=0$ has how many solutions?
a) none b) exactly one c) exactly two d) exactly four

4. Only one of the choices below is a factor of the sum $1!+2!+3!+\ldots+2011!$. Which is it? Note: $n!$ is *n factorial* and defined as $n!=n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1$.
a) 22 b) 33 c) 44 d) 55

5. You have two containers with equal volumes of liquid, one with coffee and one with tea. You then take a teaspoon of tea, pour it into the container with coffee and thoroughly mix. Then move a teaspoon of the mixture into the container of tea. Which statement is true?
a) There is more coffee in the tea container than there is tea in the coffee container.
b) There is more tea in the coffee container than there is coffee in the tea container.
c) The amount of coffee in the tea container is equal to the amount of tea in the coffee container.
d) We cannot determine which of the above statements is true without knowing the volumes.

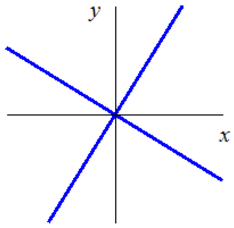
6. The geometric figure shown is composed entirely of circular arcs of radius 1. What is the area enclosed by the shape?

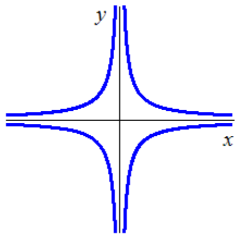


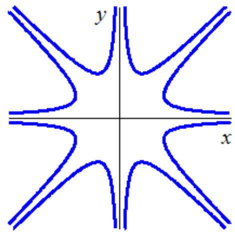
7. The exact value of $\ln\left(\tan 1^\circ\right)+\ln\left(\tan 2^\circ\right)+\ln\left(\tan 3^\circ\right)+\ldots+\ln\left(\tan 89^\circ\right)$ is

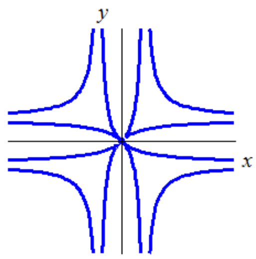
a) 0 b) 1 c) $\ln\left(\frac{\pi}{2}\right)$ d) $\ln(\pi)$

8. The graph of $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+y}$ most closely resembles which of the following?

a) 

b) 

c) 

d) 

9. If $\sec(x)-\tan(x)=2$, then what is the value of $\sec(x)+\tan(x)$?

a) -2 b) $-\frac{1}{2}$ c) $\frac{1}{2}$ d) 3

10. In a certain monogamous town, two-thirds of the men are married to three-fifths of the women. There are more than 100 men in the town. What is the least possible number of women in the town?

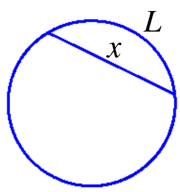
11. Suppose two ordinary 6-sided dice are colored, each with two red, two white, and two blue faces. What is the probability of rolling them so that both dice show matching colors?

12. What is the value of the expression $\frac{1}{\log_2 2011!}+\frac{1}{\log_3 2011!}+\frac{1}{\log_4 2011!}+\ldots+\frac{1}{\log_{2011} 2011!}$?

a) $\log_{2011!} 2011$ b) $1-\log_{2011!} 2011$ c) 1 d) $1+\log_{2011!} 2011$

13. Larry is rowing upstream, i.e. against the flow. As he passes under a bridge, he accidentally drops his hat in the water, but doesn't realize it until he has rowed for 5 more minutes. He then turns around and rows downstream, retrieving his hat one mile beyond the bridge. What is the rate of the stream (in miles/hour)? Assume both Larry and the stream move at constant rates, and the time it takes him to turn around is negligible.

14. A chord of length x is drawn in a circle of radius 1. The chord cuts the circumference into two arcs, as shown. The arc length, L, of the smaller arc is given by

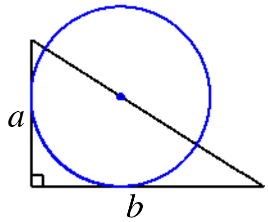


a) $\frac{\pi}{2}x$ b) $4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)$ c) $x\cos^{-1}\left(\frac{\sqrt{4-x^2}}{4}\right)$ d) $2\sin^{-1}\left(\frac{x}{2}\right)$

15. A book is missing some consecutive pages. If the sum of the page numbers of the missing pages is 355, then what is the *maximum* number of pages that can be missing? Hint: $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$

16. Suppose a drawer contains some black socks and some blue socks. If two socks are randomly chosen from the drawer, the probability they are both blue is $\frac{1}{2}$. If there are an even number of black socks, then what is the *minimum* number of blue socks in the drawer?

17. A right triangle with a height of length a and base of length b, and a circle are drawn so the circle is tangent to both legs of the triangle and its center is on the hypotenuse of the triangle, as shown. Express the radius of the circle in terms of a and b.



18. On a sunny morning, a grocer places 100 pounds of cucumbers in cases in front of his store. At that moment, the cucumbers are 99% water. Later in the day it becomes hot and dry, as a result, the cucumbers dry out a bit. At the end of the day, the grocer has not sold a single cucumber, and the cucumbers are now only 98% water. How many pounds of cucumbers remain at the end of the day?

19. Julie and Dianna have a race. Dianna gives Julie a 5 meter head-start and wins by 5 seconds. They decide to have another race, with Julie getting a 50 meter head-start, but now Dianna loses by 5 seconds. Assuming all other conditions are the same for both races (location of the finish line and average speed of each runner), then what is Julie's average speed?
a) 4.5 meters/second b) 5.5 meters/second c) 11 meters/second
d) It cannot be determined without knowing the distance of the race.

20. Either Tim, his wife, their son, or Tim's mother is a mathematician and another is an engineer.
I. If the engineer is a male, then the mathematician is a male.
II. If the mathematician is younger than the engineer, then the mathematician and the engineer are not blood relatives.
III. If the mathematician is a female, then she and the engineer are blood relatives.

Who is the mathematician?
a) Tim b) Tim's wife c) Tim's son d) Tim's mother

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New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2011 ~ Solutions

1. $A(1,2)=A(0,A(1,1))=A\big(0,A\big(0,A(1,0)\big)\big)=A\big(0,A\big(0,A(0,1)\big)\big)=A\big(0,A(0,2)\big)=A(0,3)=4$ **Answer: 4**

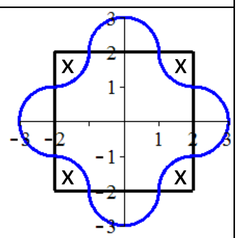
2. If the parabola formed by $f(x)=3x^2+kx+1$ is tangent to the x-axis, then it has only one (repeated) real root. Hence, the discriminant (i.e. b^2-4ac) must be zero. Thus, $k^2-4\cdot3\cdot1=0 \;\Rightarrow\; k^2=12 \;\Rightarrow\; k=\pm\sqrt{12}$, with $x=\dfrac{-k}{2\cdot3}$. Since the problem requires the point of tangency to be on the *positive* x-axis, k must be negative. Thus, $k=-\sqrt{12}=-2\sqrt{3}$. **Answer: $-\sqrt{12}$ or $-2\sqrt{3}$**

3. Completing the square for both equations:
$x^2+y^2-2x-2y+1=0 \;\Rightarrow\; (x-1)^2+(y-1)^2=1 \;\Rightarrow\;$ circle of radius 1 centered at (1,1)
$x^2+y^2-8x-10y+25=0 \;\Rightarrow\; (x-4)^2+(y-5)^2=16 \;\Rightarrow\;$ circle of radius 4 centered at (4,5)
The distance between centers is $\sqrt{(4-1)^2+(5-1)^2}=\sqrt{25}=5$, which is the sum of both radii. Hence, the circles are tangent to each other and have exactly one point of intersection. **Answer: b**

4. $1!+2!+3!+\ldots+2011!=1+2+6+24+120+\cdots$ (all remaining terms have a zero in the units position) Hence, the sum ends with a 3 in the units position. Therefore, the sum cannot have an even number as a factor, nor can 55 be a factor (since it does not end with a 0 or 5). Only 33 can be a factor. **Answer: b**

5. Let V be the volume (in teaspoons) of the containers. After pouring a teaspoon of tea in the coffee, the concentration of tea in the coffee container is $\frac{1}{V+1}$. So, the amount of tea remaining in the coffee container (after taking a teaspoon of the mixture out) is $V\left(\frac{1}{V+1}\right)=\frac{V}{V+1}$. The amount of coffee in the teaspoon of mixture (which is poured into the tea container) is $1-\frac{1}{V+1}=\frac{V}{V+1}$. Thus, the amounts are equal. **Answer: c**

6. Drawing x and y-axes and a square over the figure, as shown, helps illustrate the areas involved. The four regions that are x'ed are quarter-circles and are not included in the area, while the four semi-circles outside the square are included. Thus, the area of the region is the area of the 4×4 square (16), minus the four quarter-circle areas $(4\cdot\frac{\pi}{4}=\pi)$, plus the four semi-circle areas $(4\cdot\frac{\pi}{2}=2\pi)$. Hence, the area is $16-\pi+2\pi=16+\pi$. **Answer: $16+\pi$**



7. $\ln(\tan 1^\circ)+\ln(\tan 2^\circ)+\ln(\tan 3^\circ)+\ldots+\ln(\tan 89^\circ)=\ln\left(\frac{\sin 1^\circ}{\cos 1^\circ}\right)+\ln\left(\frac{\sin 2^\circ}{\cos 2^\circ}\right)+\ln\left(\frac{\sin 3^\circ}{\cos 3^\circ}\right)+\ldots+\ln\left(\frac{\sin 89^\circ}{\cos 89^\circ}\right)$

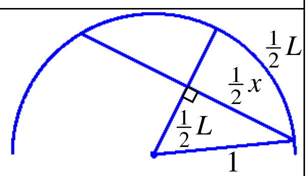
$=\ln(\sin 1^\circ)-\ln(\cos 1^\circ)+\ln(\sin 2^\circ)-\ln(\cos 2^\circ)+\ln(\sin 3^\circ)-\ln(\cos 3^\circ)+\ldots+\ln(\sin 89^\circ)-\ln(\cos 89^\circ)$

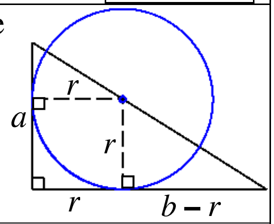
$=\left[\ln(\sin 1^\circ)+\ln(\sin 2^\circ)+\ln(\sin 3^\circ)+\ldots+\ln(\sin 89^\circ)\right]-\left[\ln(\cos 1^\circ)+\ln(\cos 2^\circ)+\ln(\cos 3^\circ)+\ldots+\ln(\cos 89^\circ)\right]$

$=\left[\ln(\sin 1^\circ)+\ln(\sin 2^\circ)+\ln(\sin 3^\circ)+\ldots+\ln(\sin 89^\circ)\right]-\left[\ln(\sin 89^\circ)+\ln(\sin 88^\circ)+\ln(\sin 87^\circ)+\ldots+\ln(\sin 1^\circ)\right]^*$

$=0$

*Since $\cos(x^\circ)=\sin(90^\circ-x^\circ)$ **Answer: a**

8.	Multiplying by the LCD $xy(x+y)$, provided $x\neq 0$, $y\neq 0$, and $x+y\neq 0$, gives: $y(x+y)-x(x+y)=xy \Rightarrow yx+y^2-x^2-xy=xy \Rightarrow y^2-xy-x^2=0$ Solving for *y*, using the quadratic formula gives: $y=\left(\frac{1\pm\sqrt{5}}{2}\right)x$, i.e. two lines through the origin (open at the origin). Answer: a
9.	Multiplying by the conjugate $\sec(x)+\tan(x)$ gives: $[\sec(x)-\tan(x)][\sec(x)+\tan(x)]=2[\sec(x)+\tan(x)]$ $\Rightarrow \sec^2(x)-\tan^2(x)=2[\sec(x)+\tan(x)]$, the identity $1+\tan^2(x)=\sec^2(x) \Rightarrow 1=\sec^2-\tan^2(x)$ $\Rightarrow 1=2[\sec(x)+\tan(x)] \Rightarrow \sec(x)+\tan(x)=\frac{1}{2}$ Answer: c
10.	Let *m* = the number of men, and *w* = the number of women in the town $\Rightarrow \frac{2}{3}m=\frac{3}{5}w \Rightarrow w=\frac{10}{9}m$, thus *m* must be a multiple of 9...the smallest integer greater than or equal to 100 that is a multiple of 9 is 108 $\Rightarrow w=\frac{10}{9}(108)=120$ Answer: 120
11.	For a matching color, the second die must match the first. The first die can show any color, red say. Then the probability the second die matches is $\frac{2}{6}=\frac{1}{3}$. Answer: $\frac{1}{3}$
12.	$\frac{1}{\log_2 2011!}+\frac{1}{\log_3 2011!}+\ldots+\frac{1}{\log_{2011} 2012!}=\frac{1}{\left(\frac{\log 2011!}{\log 2}\right)}+\frac{1}{\left(\frac{\log 2011!}{\log 3}\right)}+\ldots+\frac{1}{\left(\frac{\log 2011!}{\log 2011}\right)}$ $=\frac{\log 2}{\log 2011!}+\frac{\log 3}{\log 2011!}+\ldots+\frac{\log 2011}{\log 2011!}=\frac{\log 2+\log 3+\ldots+\log 2011}{\log 2011!}=\frac{\log(2\cdot 3\cdot\ldots\cdot 2011)}{\log 2011!}$ $=\frac{\log 2011!}{\log 2011!}=1$ Answer: c
13.	Let *x* be Larry's rowing speed, *y* the speed of the stream, and *t* the time (in minutes) he takes to row back to reach the hat. Thus, we get ❶ $(x+y)t=(x-y)\cdot 5+1$, the distance traveled back down stream to retrieve the hat equals the distance traveled from the turn-around point to the bridge plus 1 mile, and ❷ $y(5+t)=1$, the distance the hat traveled downstream. Expanding ❶ gives $xt+yt=5x-5y+1$, now substituting $1=5y+yt$ from ❷ $\Rightarrow xt+yt=5x-5y+5y+yt \Rightarrow xt=5x \Rightarrow t=5$ Substituting back into ❷ gives $10y=1 \Rightarrow y=\frac{1}{10}$ mile/minute, which is 6 miles/hour. Answer: 6
14.	Drawing a radius as a perpendicular bisector of the chord and the arc, forms a right triangle with a hypotenuse of 1 and a base of $\frac{1}{2}x$. The central angle, in radians, equals the subtended arc (since the radius of the circle is 1). Hence, $\sin\left(\frac{L}{2}\right)=\frac{\left(\frac{x}{2}\right)}{1} \Rightarrow \sin\left(\frac{L}{2}\right)=\frac{x}{2} \Rightarrow \frac{L}{2}=\sin^{-1}\left(\frac{x}{2}\right) \Rightarrow 2\sin^{-1}\left(\frac{x}{2}\right)$  Answer: d
15.	Let *m* be the first numbered page that is missing, and *n* be the last numbered page that is missing. The sum of all the missing pages is then $(1+2+3+\cdots+(m-1)+m+\cdots+n)-(1+2+3+\cdots+(m-1))$ $=\frac{1}{2}n(n+1)-\frac{1}{2}(m-1)(m-1+1)=\frac{1}{2}\left[(n^2+n)-(m^2-m)\right]=\frac{1}{2}\left[(n^2-m^2)+(n+m)\right]$ $=\frac{1}{2}[(n+m)(n-m)+(n+m)]=\frac{1}{2}(n+m)[(n-m)+1]=\frac{1}{2}(n+m)(n-m+1)$, which must equal 355 $\Rightarrow (n+m)(n-m+1)=710$ The factorization of 710 is $1\cdot 2\cdot 5\cdot 71$. We want the maximum number of pages, so we seek to maximize $n-m$. Also, since $m\geq 1$, $n+m>n-m+1$. Thus, $n+m=71$ and $n-m+1=1\cdot 2\cdot 5$. Solving gives $m=31$ and $n=40$, which is 10 missing pages. Answer: 10

16.	Let m = the number of black socks (an even number), and n = the number of blue ones. The probability of getting two blue socks is $\frac{n}{n+m}\cdot\frac{n-1}{n+m-1}=\frac{1}{2}$. Expanding, cross-multiplying, and combining like terms (for n) gives: $n^2-(1+2m)n+m-m^2=0$. If $m=2$, then $n^2-5n-2=0$, which has no positive integer solution. If $m=4$, then $n^2-9n-12=0$, which also has no positive integer solution. If $m=6$, then $n^2-13n-30=0$, which factors as: $(n-15)(n+2)=0$. Thus, $n=15$. We know this is the smallest solution, since as m increases so does n. Answer: 15
17.	Draw two radii that are perpendicular to the legs of the triangle, as shown, then use similar triangles to obtain the desired relationship. Hence, $\frac{a}{b}=\frac{r}{b-r}\Rightarrow br=ab-ar$ $\Rightarrow ar+br=ab \Rightarrow r=\dfrac{ab}{a+b}$. Answer: $\dfrac{ab}{a+b}$ 
18.	If 100 pounds are 99% water, then 99 pounds are water with only 1 pound of flesh. After dehydration, some water evaporates but we still have 1 pound of flesh — which now represents 2% of what remains (since 98% is water). Letting x represent the number of pounds remaining, we get $\dfrac{1}{x}=0.02 \Rightarrow \dfrac{1}{x}=\dfrac{2}{100}=\dfrac{1}{50}\Rightarrow x=50$ Answer: 50
19.	In the first race Julie runs $50-5=45$ meters further than in the second race, and requires an additional $5+5=10$ seconds to do it. Hence, her speed is $\frac{45\text{ meters}}{10\text{ seconds}}=4.5$ meters/sec. Answer: a
20.	Tim's wife and mother are not blood relatives. So from III, if the mathematician is a female, the engineer is a male. But from I, if the engineer is a male, then the mathematician is a male. Thus, there is a contradiction, if the mathematician is a female. Hence, either Tim or his son is the mathematician. Tim's son is the youngest of all four and is a blood relative of each of them. So from II, Tim's son is not the mathematician. Hence, Tim is the mathematician. Answer: a Note: From II, Tim's mother cannot be the mathematician. So the engineer is either his wife or his son. It is not possible to determine anything further.

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