New York State Mathematics
Association of Two-Year Colleges

Math League Contest ~ Fall 2015

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four (4) points are awarded for each correct answer, one (1) point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc.

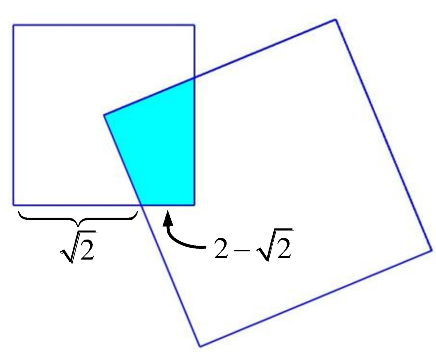
1. If we define $x @ y$ to be the average of x and y, i.e. $x @ y = \frac{x+y}{2}$, then which of the following "distributive laws" are true for all real numbers x, y, and z?
   I. $x @ (y+z) = (x @ y) + (x @ z)$
   II. $x + (y @ z) = (x+y) @ (x+z)$
   III. $x @ (y @ z) = (x @ y) @ (x @ z)$
   A) I only   B) II only   C) III only   D) II and III only

2. A 4-digit number, *WXYZ*, in which *W*, *X*, *Y*, and *Z* each represent a different digit, is formed according to the following three rules: ① $X = W + Y + Z$ ② $W = Y + 1$ ③ $Z = W - 5$
   What is the four-digit number?

3. When a list of numbers is expanded to include the number 25, the mean is increased by 2. When the expanded list is further expanded to also include the number 1, the new mean decreases by 2, giving the mean of the original list. How many numbers were in the original list?

4. How many *different pairs* of prime numbers have a sum of 2015?

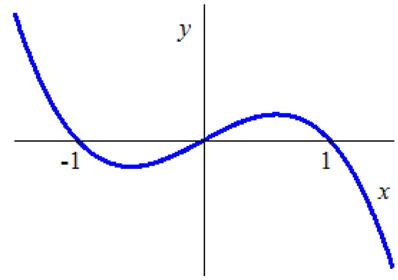
5. Two squares, one with side lengths of 2 and one with a side lengths of 3, overlap so that a corner of the larger square is at the center of the smaller square. Furthermore, a side of the larger square cuts the base of the smaller square into parts of length $\sqrt{2}$ and $2-\sqrt{2}$, as shown. What is the area of the overlap (i.e. the shaded region)?



6. What is the remainder when the polynomial $x^{2016} + x^{2015} + x^{2014} + 2015$ is divided by $x+1$?
   A) 0   B) 2014   C) 2015   D) 2016

7. How many perfect squares factor $10!$? Note: $n! = n(n-1)(n-2)\cdots 2\cdot 1$, e.g. $3! = 3\cdot 2\cdot 1 = 6$.

8. The diagram shows part of the graph of $y = ax^3 + bx^2 + cx + d$, with roots at -1, 0, and 1. Which of the following statements *must* be true? I. $a < 0$ II. $b = 0$ III. $c > 0$

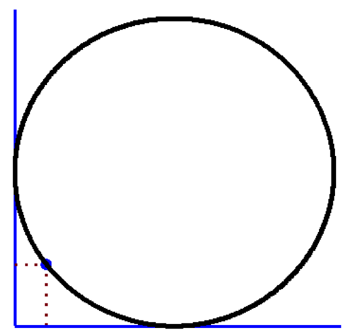


A) I only B) I and III only
C) I and II only D) I, II, and III

9. Suppose you have a box of 100 keys, only one of which opens a lock that you need to unlock. You select one key at random and try it. If it does not open the lock, you discard it and try another key at random from the remaining keys. You proceed in this way until you obtain the correct key. What is the probability you find the correct key on your 50th attempt? Assume each key in the box is equally likely to be chosen.

A) $\frac{1}{50!}$ B) $\frac{1}{100}$ C) $\frac{49}{100}$ D) $\frac{1}{2}$

10. A large circular table is pushed to the corner of a room so that it is touching the two walls (the two walls are perpendicular to each other). There is a point on the circumference of the table that is exactly 1 foot from one wall and 2 feet from the other wall, as shown. What is the *diameter* of the table (in feet)?



11. The expression $\sqrt{3-\sqrt{8}}$ is called a *nested radical* (i.e. a radical or radicals within a radical). Writing a nested radical as the sum or difference of simple radicals, is called *denesting*. An integer is said to be *square-free* if no perfect square other than 1 is a factor (e.g. 10 is square-free, but 12 is not since 4 is a factor). $\sqrt{3-\sqrt{8}}$ can be denested to the form $a\sqrt{b}+c$, where a, b, and c are integers, and b is square-free. What is the sum $a+b+c$?

12. A *googol* is defined as 10^{100}, i.e. a 1 followed by 100 zeros. A *googolplex* is defined as 10^{googol}, i.e. a 1 followed by a googol zeros. $\log_{\text{googol}}(\text{googolplex}) = 10^n$, for some integer n. What is n?

13. $\arctan(2015) + \arctan\left(\frac{1}{2015}\right) = ?$

A) $\frac{\pi}{4}$ B) $\frac{1007}{2015}\pi$ C) $\frac{\pi}{2}$ D) $\frac{1008}{2015}\pi$

14. The equation $\dfrac{1}{x^2-2x+1} + \dfrac{2}{x^2-2x+3} = 1$ has two real solutions and two complex solutions. What is the numerical value of the product of the two real solutions?

15. $\dfrac{(1-i)^{2016}}{(1+i)^{2014}} = ?$ (Where $i \equiv \sqrt{-1}$.)

A) $-2i$ B) $2i$ C) $2-2i$ D) $2+2i$

16. The pages of books and magazines are numbered so that when open, even numbered pages are on the left and odd numbered pages are on the right. I have a magazine where one page has been removed at random, so that now two page numbers are missing. If the sum of the remaining pages is 832, what is the last numbered page? Assume, of course, that pages are consecutively numbered, starting with the number 1. Hint: The sum of the first n counting numbers is $\frac{1}{2}n(n+1)$.

17. What is the *maximum* value of the function $f(x) = \cos^4(x) + \sin^2(2x)$?

A) $\dfrac{5}{4}$ B) $\dfrac{21}{16}$ C) $\dfrac{4}{3}$ D) $\dfrac{3}{2}$

18. The equation $\log_x(2) - \log_2(x) = 2$ has two solutions for x, one is less than 1 and one greater than 1. What is the solution that is greater than 1?

19. If a and b are single-digit positive integers chosen at random (so that each digit has an equal chance of being selected), then what is the probability the point (a, b) is *on* the line $y = ax - b$?

A) 0 B) $\dfrac{1}{81}$ C) $\dfrac{2}{81}$ D) $\dfrac{ab}{81}$

20. A 24-hour clock, i.e. one that shows AM and PM times, loses 75 seconds per day and currently shows the correct time. At this rate, and unadjusted, how many years (rounded to the nearest year) will it take until the clock will next show the correct time?

A) 2 B) 3 C) 4 D) 5

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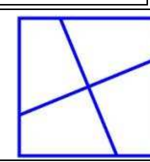
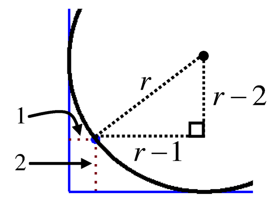
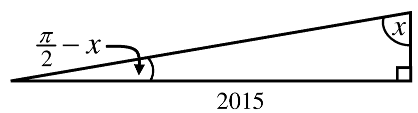
Math League Contest ~ Fall 2015 ~ Solutions

1.	I. $x@(y+z)=\frac{x+y+z}{2}$ vs $(x@y)+(x@z)=\frac{x+y}{2}+\frac{x+z}{2}=\frac{2x+y+z}{2}$, Not Equal II. $x+(y@z)=x+\frac{y+z}{2}=\frac{2x+y+z}{2}$ vs $(x+y)@(x+z)=\frac{x+y}{2}+\frac{x+z}{2}=\frac{2x+y+z}{2}$, Equal III. $x@(y@z)=x@\frac{y+z}{2}=\frac{x+\frac{y+z}{2}}{2}=\frac{2x+y+z}{4}$ vs $(x@y)@(x@z)=\frac{x+y}{2}@\frac{x+z}{2}=\frac{\frac{x+y}{2}+\frac{x+z}{2}}{2}=\frac{2x+y+z}{4}$, Equal Answer: D
2.	$W=Y+1$ and $Z=W-5 \Rightarrow Z=Y-4$; thus, $X=W+Y+Z=Y+1+Y+Y-4=3Y-3$. Since these must all be digits, $0\le 3Y-3\le 9 \Rightarrow 1\le Y\le 4$ and $0\le Y-4\le 9 \Rightarrow 4\le Y\le 13$. Thus, $Y=4, W=5, Z=0$, and $X=9$. Therefore, the 4-digit number is 5940. Answer: 5940
3.	Let n be the original list size, and S be the sum. Thus, ① $\frac{S+25}{n+1}=\frac{S}{n}+2$ and ② $\frac{S+25+1}{n+2}=\frac{S}{n}$. These simplify to: ① $2n^2-23n+S=0$ and ② $S=13n$. Combining ① and ② yields: $2n(n-5)=0$, giving $n=5$ as the only viable solution (and a mean of 13). Answer: 5
4.	2015 is odd, the only way the sum of two integers can be odd is if one is even and one is odd. The only even prime number is 2, making its pair 2013. But 2013 is not prime, as it is divisible by 3 (and 11 and 61). Thus, there are no pairs of primes that sum to 2015. Answer: None
5.	Extending the sides of the large square into the smaller square, reveals 4 congruent sections. Thus, each region is $\frac{1}{4}$ the area of the entire (small 2×2) square, $\frac{1}{4}\cdot 4=1$.  Answer: 1
6.	The Polynomial Remainder Theorem tells us that when the polynomial $P(x)$ is divided by the linear polynomial $x-a$, the remainder is $P(a)$. Thus, the remainder in this case is $P(-1)=(-1)^{2016}+(-1)^{2015}+(-1)^{2014}+2015=1-1+1+2015=2016$. Answer: D
7.	$10!=10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=2\cdot 5\cdot 3^2\cdot 2^3\cdot 7\cdot 2\cdot 3\cdot 5\cdot 2^2\cdot 3\cdot 2=2^8\cdot 3^4\cdot 5^2\cdot 7$. Any factor will thus be of the form $2^a\cdot 3^b\cdot 5^c\cdot 7^d$, where a, b, c, and d are integers such that $0\le a\le 8$, $0\le b\le 4$, $0\le c\le 2$, and $0\le d\le 1$. However, in order for such a factor to be a perfect square, a, b, c, and d must be even. Therefore, $a\in\{0,2,4,6,8\}$, $b\in\{0,2,4\}$, $c\in\{0,2\}$, $d\in\{0\}$, i.e. 5 values for a, 3 values for b, 2 values for c, and only 1 value for d. Hence, there are $5\cdot 3\cdot 2\cdot 1=30$ perfect square factors of 10!. Answer: 30
8.	Roots being $x=\pm 1$ and $x=0$, tell us $y=ax(x-1)(x+1)=ax^3-ax$. Thus, $b=0$ (and $d=0$) and $c=-a$. Furthermore, the long-term behavior ($y\to-\infty$ as $x\to\infty$, and $y\to\infty$ as $x\to-\infty$) tell us that $a<0$, hence, $c>0$. Answer: D

9.	Each key is equally likely to be the correct one, thus the probability is $\frac{1}{100}$. **Answer: B** Alternate Solution: $P = \frac{99}{100}\cdot\frac{98}{99}\cdot\frac{97}{98}\cdots\frac{51}{52}\cdot\frac{1}{51} = \frac{1}{100}$
10.	We can construct a right triangle with the radius of the table, r, as the hypotenuse and legs as shown. The Pythagorean Theorem yields: $(r-2)^2+(r-1)^2=r^2 \Rightarrow r^2-6r+5=0 \Rightarrow (r-1)(r-5)=0$. Thus, r is either 1 foot or 5 feet. However, the diagram shows the point on the circumference on the short arc subtended by the walls. Thus, $r=5$, making the diameter 10 feet.  **Answer: 10**
11.	Letting $\sqrt{3-\sqrt{8}} = a\sqrt{b}+c$, we solve for a, b, and c. First, square both sides to obtain: $3-2\sqrt{2} = a^2b+2ac\sqrt{b}+c^2$. Rearranging the right-hand-side of the last result, gives: $3-2\sqrt{2} = a^2b+c^2+2ac\sqrt{b}$. Equating corresponding terms yields: ① $3=a^2b+c^2$ and ② $-2\sqrt{2}=2ac\sqrt{b}$. Equation ② tells us that $\boxed{b=2}$ and $2ac=-2$, thus ③ $ac=-1$. Equations ① and ③ both tell us $a=\pm1$ and $c=\mp1$. Therefore, $\sqrt{3-\sqrt{8}}$ is either $-\sqrt{2}+1=1-\sqrt{2}$ or $\sqrt{2}-1$. However, $1-\sqrt{2}$ is negative, and clearly $\sqrt{3-\sqrt{8}}$ is positive. Hence, $\sqrt{3-\sqrt{8}}=\sqrt{2}-1$, so that $\boxed{a=1}$ and $\boxed{c=-1}$, giving $a+b+c=1+2+(-1)=2$. **Answer: 2**
12.	Letting $x=\log_{googol}(googolplex)$, gives: $googol^x = googolplex \Rightarrow \left(10^{100}\right)^x = 10^{10^{100}}$ $\Rightarrow 10^{100x}=10^{10^{100}} \Rightarrow 100x=10^{100}$, dividing by 100 yields: $x=10^{98}$. **Answer: 98**
13.	Let $x=\arctan(2015) \Rightarrow \tan(x)=\frac{2015}{1}=\frac{opposite}{adjacent}$, giving the triangle shown. Thus, $\tan\left(\frac{\pi}{2}-x\right)=\frac{1}{2015} \Rightarrow \arctan\left(\frac{1}{2015}\right)=\frac{\pi}{2}-x$.  This gives: $\arctan(2015)+\arctan\left(\frac{1}{2015}\right)=x+\frac{\pi}{2}-x=\frac{\pi}{2}$. **Answer: C**
14.	Letting $y=x^2-2x+1$, we can express the other denominator as $(x^2-2x+1)+2=y+2$. The equation now simplifies to $\frac{1}{y}+\frac{2}{y+2}=1$. Multiplying by the LCD, gives $y+2+2y=y(y+2)$. Expanding and collecting terms yields: $y^2-y-2=0$, which factors as $(y+1)(y-2)=0$, giving $y=-1,\ 2$. This gives the two equations for x: ① $x^2-2x+1=-1$ and ② $x^2-2x+1=2$. Solving yields: ① $x=1\pm i$ and ② $x=1\pm\sqrt{2}$. Hence, the product of the two real solutions ② is $\left(1+\sqrt{2}\right)\left(1-\sqrt{2}\right)=-1$. **Answer: −1**
15.	$(1-i)^2=-2i$ and $(1+i)^2=2i$, thus $\frac{(1-i)^{2016}}{(1+i)^{2014}} = \frac{\left((1-i)^2\right)^{1008}}{\left((1+i)^2\right)^{1007}} = \frac{(-2i)^{1008}}{(2i)^{1007}} = \frac{2^{1008}i^{1008}}{2^{1007}i^{1007}} = 2^1 i^1 = 2i$ **Answer: B**

16.	Let p be the first page (odd) number of the missing page, then $p+1$ (even) is the page number on the back side of the torn-out page, giving a sum of $2p+1$ (odd). Let n be the last numbered page in the magazine. The sum of the remaining pages is 832 tells us: $1+2+3+\ldots+(n-1)+n-(2p+1)=832$, or $1+2+3+\ldots+(n-1)+n=832+(2p+1)$. Using the formula $1+2+3+\ldots+(n-1)+n=\frac{1}{2}n(n+1)$, gives: $\frac{1}{2}n(n+1)=832+(2p+1)$, where $1\le p\le n-1$. To obtain an approximation (a lower bound) for *n*, solve $\frac{1}{2}n(n+1)\approx 832$. Thus, $n(n+1)\approx 1664$ or $n\approx 40$ (since $40^2=1600$). Since $2p+1$ is odd, the sum $1+2+3+\ldots+(n-1)+n=\frac{1}{2}n(n+1)$ must also be odd, so the difference is even (832). That means neither n or $n+1$ can be a multiple of 4. Suppose $n=45$, then the sum of all 45 page numbers is $\frac{1}{2}\cdot 45\cdot 46>1000$, too large for the removal of any two page numbers to reduce the sum to 832. Thus, n must be 41 or 42. If $n=41$, then the sum of all page numbers is $\frac{1}{2}\cdot 41\cdot 42=861$, making the two missing page numbers 30 and 31. However, the lowered numbered page must be odd. Thus, n must be 42, so the sum of all pages is $\frac{1}{2}\cdot 42\cdot 43=903$ (and the torn-out page is numbered 35 and 36). **Answer: 42**
17.	Using the double-angle formula: $\sin(2x)=2\cos(x)\sin(x)$, we get: $f(x)=\cos^4(x)+\sin^2(2x)=\cos^4(x)+\left(2\cos(x)\sin(x)\right)^2=\cos^4(x)+4\cos^2(x)\sin^2(x)$, and now using $\sin^2(x)=1-\cos^2(x)$ gives: $f(x)=\cos^4(x)+4\cos^2(x)\left(1-\cos^2(x)\right)$. Simplifying this last result and completing-the-square, gives: $f(x)=-3\cos^4(x)+4\cos^2(x)$, and then $f(x)=-3\left[\cos^4(x)-\frac{4}{3}\cos^2(x)+(-\frac{2}{3})^2-(-\frac{2}{3})^2\right]=-3\left[\left(\cos^2(x)-\frac{2}{3}\right)^2-\frac{4}{9}\right]=-3\left(\cos^2(x)-\frac{2}{3}\right)^2+\frac{4}{3}$. Thus, f is maximized when $\cos^2(x)-\frac{2}{3}=0$, giving $f_{\max}=-3(0)+\frac{4}{3}=\frac{4}{3}$. **Answer: C**
18.	Letting $y=\log_2(x)=\dfrac{\log(x)}{\log(2)}$. Thus, $\log_x(2)=\dfrac{\log(2)}{\log(x)}=\dfrac{1}{y}$. Now the equation can be expressed as: $\dfrac{1}{y}-y=2 \Rightarrow y^2+2y-1=0 \Rightarrow y=\dfrac{-2\pm\sqrt{4+4}}{2}=-1\pm\sqrt{2}$. Thus, $x=2^{-1-\sqrt{2}}$ or $x=2^{-1+\sqrt{2}}$. However, $2^{-1-\sqrt{2}}<1$, while $2^{-1+\sqrt{2}}=2^{\sqrt{2}-1}>1$. Therefore, $x=2^{\sqrt{2}-1}$. **Answer: $2^{\sqrt{2}-1}$**
19.	Since a and b are both integers from the set $\{1,2,3,4,5,6,7,8,9\}$, there are $9^2=81$ ordered pairs of the form (a,b). In order for (a,b) to be *on* the line $y=ax-b$, we need $b=a\cdot a-b$. Thus, we need $2b=a^2$ or $b=\frac{1}{2}a^2$. Now we simply count how many of the 81 possible ordered pairs (a,b) satisfy that equation. The only values of a for which $\frac{1}{2}a^2$ is one of the digits, is $a=2$ and $a=4$ (giving $b=2$ and $b=8$, respectively). Thus, only 2 ordered pairs, $(2,2)$ and $(4,8)$, out of 81 satisfy the condition. Hence, the probability is $\frac{2}{81}$. **Answer: C**
20.	75 seconds = 1.25 minutes. 1.25 min/day × 360 days (approx. days per year) = 450 min/year, which equals $\frac{450}{60}=7.5$ hrs/year. The clock will next show the correct time after 24 hrs have been lost: $\frac{24\text{ hrs}}{7.5\text{ hrs/yr}}\approx 3$ yrs. **Answer: B**