NYS MATYC 1967

New York State Mathematics Association of Two-Year Colleges

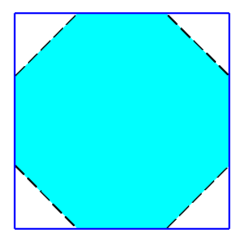
Math League Contest ~ Fall 2016

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. Let $f^n(x)$ denote the n^{th} iteration of function f, i.e. $f^n(x) = \underbrace{f(f(f\cdots f}_{n}(x)\cdots))$. For example, $f^2(x) = f(f(x))$ and $f^3(x) = f(f(f(x)))$. If $f(x) = 1 - \frac{1}{x}$, then which of the following represents $f^{2016}(x)$ when simplified?

   A) x B) $\frac{1}{1-x}$ C) $1-\frac{1}{x}$ D) NOTA

2. Suppose a regular octagon (i.e. an 8-sided polygon with congruent sides and interior angles) is to be constructed from a 3×3 square by removing 4 triangular corners, as shown. What will be the length of each edge of the octagon?



3. How many real values for x solve the equation $x|x| + \frac{x}{|x|} = 5x + 4$? Note: $|x|$ is the absolute value of x.

   A) 1 B) 2 C) 3 D) 4

4. What is the remainder when the 2016 digit number $\underbrace{201620162016\ldots2016}_{503 \text{ groups of } 2016}2017$, i.e. the number consisting of 503 groups of 2016 followed by 2017, is divided by 7?

   A) 0 B) 1 C) 2 D) 6

5. If the positive integers are arranged in a triangular pattern, as shown, then "2016" will eventually appear in the m^{th} row and n^{th} position from the left. What is $m+n$? (e.g. "12" appears in the 5th row and 2nd position from the left, thus $m=5$ and $n=2$, making $m+n=7$.)
   Hint: $1+2+3+\ldots+k = \frac{1}{2}k(k+1)$.

```
                1
            2       3
        4       5       6
    7       8       9       10
11      12      13      14      15
    ⋮       ⋮       ⋮       ⋮
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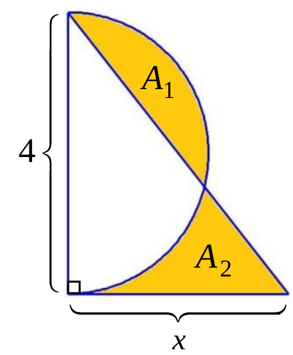
6. The equation $\log_{2016}(x)+\log_{x}(2016)=\frac{5}{2}$ has two real solutions, x_1 and x_2. What is the product, $x_1 \cdot x_2$, of these two solutions?

A) $2016^{2/5}$ B) $2016^{\sqrt{5}}$ C) $2016^{5/2}$ D) $2016^{\sqrt{10}}$

7. A right triangle with a height of 4 and unknown base, x, intersects a semicircle whose diameter coincides with the height of the triangle, as shown. A_1 is the area outside the triangle, but inside the semicircle; while A_2 is the area inside the triangle, but outside the semicircle. If $A_1 = A_2$, then what is x?



A) 3 B) π C) $\sqrt{10}$ D) $\frac{10}{\pi}$

8. A *lattice point* in the *xy*-plane is a point with integer coordinates. How many lattice points fall on the line $y=\frac{2}{5}x+\frac{2}{7}$?

A) none B) exactly one C) exactly two D) infinitely many

9. If the line $y=x$ is rotated 75° counter-clockwise about the point $(1,1)$, what will be the *y*-intercept of the new line?

10. A box initially contains 10 marbles, 7 red and 3 blue. If 9 marbles are randomly selected and removed from the box, what is the probability that the remaining marble is red?

11. $\arccos\left(\frac{1}{\pi}\right)+\arcsin\left(\frac{1}{\pi}\right)=?$

A) $\frac{2}{\pi}$ B) 1 C) $\frac{\pi}{2}$ D) π

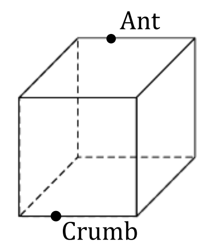
12. If $\cos(x)=\cos(50°)+\cos(70°)$, then x is? Hint: $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$

A) 0° B) 10° C) 20° D) 30°

13. Which value for x solves the equation $9^x-6^x=2\cdot 4^x$?

A) $\frac{\ln(2)}{\ln(3)-\ln(2)}$ B) $\frac{\ln(3)}{\ln(3)-\ln(2)}$ C) $\frac{\ln(9)}{\ln(2)\cdot\ln(3)}$ D) $\frac{\ln(9)}{\ln(2)}$

14. A cube with edges of length 3, has an ant and a crumb both 1 unit from the nearest corner on opposite edges, as shown. What is the shortest distance the ant can traverse to get to the crumb? Note: The ant cannot jump, and must remain in contact with the cube at all times.

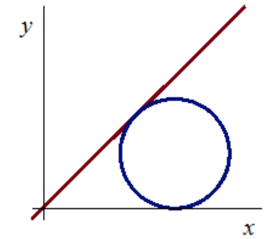


15. As of September 2016 the largest known prime number is $2^{74,207,281}-1$. What are the last two digits in its expansion? For example, $2^{10}=1024$, so the last two digits are 24.

16. The *Lambert W* function is defined as the inverse function of $f(x)=xe^x$ for $x\geq -1$, i.e. $W(x)=f^{-1}(x)$ for $x\geq -e^{-1}$. Thus, if $ye^y=x$, then $y=W(x)$. If $x^2e^x=4$, then x equals

A) $\frac{1}{2}W(1)$ B) $\frac{1}{2}W(2)$ C) $2W(1)$ D) $2W(2)$

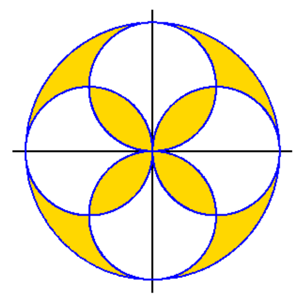
17. A circle of radius 1 is placed in the first quadrant so that it is tangent to the line $y=x$ and the x-axis, as shown. What is the x-coordinate of the point of tangency on the x-axis?



18. If x is the smallest positive angle that satisfies the equation: $\cos(x)+\sin(x)=\frac{7}{5}$, then what is the value of $\tan(2x)$?

A) $\frac{49}{25}$ B) $\frac{49}{24}$ C) $\frac{24}{7}$ D) $\frac{25}{7}$

19. The diagram shows a large circle of radius 1 with four smaller circles of radius ½ that are tangent to the larger circle with centers along the coordinate axes. What is the area of the shaded region?



20. The seven cards shown have a letter on one side and a number on the other. How many of the cards <u>must</u> be turned over in order to verify the statement: *If a card has a vowel on one side, then the other side must have an odd number.*?

A	**B**	**E**	**2**	**0**	**1**	**6**

A) 3 B) 4 C) 5 D) 6

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New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2016 ~ Solutions

1.

$f^1(x) = f(x) = 1 - \frac{1}{x} \Rightarrow f^2(x) = f\left(f(x)\right) = f\left(1 - \frac{1}{x}\right) = 1 - \frac{1}{\left(1 - \frac{1}{x}\right)} = 1 - \frac{x}{x-1} = \frac{1}{1-x}$

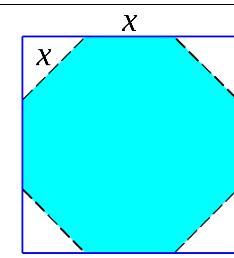
$\Rightarrow f^3(x) = f\left(f^2(x)\right) = f\left(\frac{1}{1-x}\right) = 1 - \frac{1}{\left(\frac{1}{1-x}\right)} = 1 - (1-x) = x \Rightarrow f^4(x) = f\left(f^3(x)\right) = f(x) = f^1(x).$

Thus, $f^n(x)$ cycles by 3: $f^1(x) = f^4(x) = f^7(x) = \ldots = f^{3k+1}(x)$, and $f^2(x) = f^5(x) = \ldots = f^{3k+2}$, and $f^3(x) = f^6(x) = \ldots = f^{3k}$. Since $2016 = 3 \cdot 672$, $f^{2016}(x) = f^{3\cdot 672}(x) = x$.

Answer: A

2.

Let x = the length of each edge. Since each corner is a 45°-45°-90° triangle, with a hypotenuse of length x, the leg length is $\frac{\sqrt{2}}{2}x$. Thus, each side of the square has length $\frac{\sqrt{2}}{2}x + x + \frac{\sqrt{2}}{2}x = \left(1+\sqrt{2}\right)x = 3$. Solving for x gives: $x = \frac{3}{1+\sqrt{2}} = 3\left(\sqrt{2}-1\right)$.



Answer: $\frac{3}{1+\sqrt{2}} = 3\left(\sqrt{2}-1\right)$

3.

Case I: $x > 0$: $x^2 + 1 = 5x + 4 \Rightarrow x^2 - 5x - 3 = 0 \Rightarrow x = \frac{5 \pm \sqrt{37}}{2}$, but $\frac{5 - \sqrt{37}}{2} < 0$.

Thus, only $x = \frac{5+\sqrt{37}}{2}$ is a solution in this case.

Case II: $x < 0$: $-x^2 - 1 = 5x + 4 \Rightarrow x^2 + 5x + 5 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{5}}{2}$, both satisfy $x < 0$.

Therefore, there are exactly 3 solutions.

Answer: C

4.

Since 2016 is a multiple of 7 (2016 = 7·288), and 20162016...2017 = 20162016...2016+1 = 7·02880288...0288+1, the remainder must be 1.

Answer: B

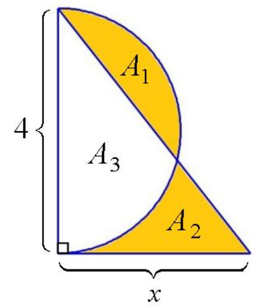
5.

Notice that the last number in row k, called triangular numbers, is $1+2+3+\ldots+k$. For example, the 6 in row 3 equals $1+2+3$ and the 10 in row 4 equals $1+2+3+4$. Thus, we seek the largest value of k such that $1+2+3+\ldots+k \le 2016$ in order to identify the row preceding 2016 (or *the* row if the sum = 2016). Using the formula for the sum, we need to determine the largest k-value for which: $\frac{1}{2}k(k+1) \le 2016$ or $k(k+1) \le 4032$. Since $60^2 = 3600$, we know k will be in the low 60's. $60 \cdot 61 = 3660$, $61 \cdot 62 = 3782$, $62 \cdot 63 = 3906$, and $63 \cdot 64 = 4032$, giving a k-value of 63 with 2016 being the last number of that row (since the sum = 2016). Thus, 2016 appears in the 63rd row and the 63rd position, making $m + n = 63 + 63 = 126$.

Answer: 126

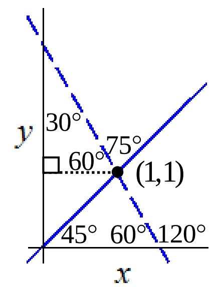
6. Since $\log_{2016}(x)=\dfrac{\log(x)}{\log(2016)}$ and $\log_x(2016)=\dfrac{\log(2016)}{\log(x)}$, let $y=\log_{2016}(x)$, then $y^{-1}=\log_x(2016)$. Now we solve $y+y^{-1}=\dfrac{5}{2}$ or $y^2-\dfrac{5}{2}y+1=0$. We know the sum of the two solutions of the quadratic is $-\left(-\dfrac{5}{2}\right)=\dfrac{5}{2}$. Thus, $y_1+y_2=\dfrac{5}{2}$ or $\log_{2016}(x_1)+\log_{2016}(x_2)=\dfrac{5}{2}$, which gives $\log_{2016}(x_1x_2)=\dfrac{5}{2}$, and finally $x_1x_2=2016^{5/2}$. Answer: C

7. Let the unshaded region (common to both the triangle and semicircle) be A_3. Thus, the area of the semicircle is $A_S=A_1+A_3=2\pi$, and the area of the triangle is $A_T=A_2+A_3=2x$. Since $A_1=A_2$, $A_S=A_T$, which gives $2x=2\pi$ or $x=\pi$. Answer: B



8. $y=\frac{2}{5}x+\frac{2}{7}\ \Rightarrow\ 5y-2x=\frac{10}{7}$, which has no integer solutions. Answer: A

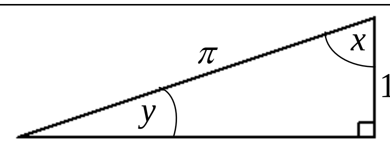
9. The 75° rotation of the line $y=x$ (solid) makes the new line (dashed) form angles of 120° (45° + 75°) and 60° with the x-axis, as shown. Thus, we see two 30°-60°-90° triangles formed. The smaller one has a base (dotted) of length 1 and height $\sqrt{3}$. Therefore, the y-intercept of the new line is $\sqrt{3}+1$. Answer: $\sqrt{3}+1$



Alternate Solution: The slope of a line equals $\tan(\theta)$, where θ is the angle of inclination, 120° in this case. Thus, the slope is $\tan(120°)=-\tan(60°)=-\sqrt{3}$, and the equation is: $y-1=-\sqrt{3}(x-1)$ or $y=-\sqrt{3}x+\sqrt{3}+1$. Hence, the y-intercept is $\sqrt{3}+1$.

10. The probability the remaining marble is red, is the same as the probability of selecting a red one from the 10 marbles. Thus, the probability is $\frac{7}{10}$. Answer: $\frac{7}{10}$

11. Letting $x=\arccos\left(\frac{1}{\pi}\right)$ and $y=\arcsin\left(\frac{1}{\pi}\right)$, gives $\cos(x)=\frac{1}{\pi}$ and $\sin(y)=\frac{1}{\pi}$. A right triangle corresponding to these values is shown. Thus, $x+y=\frac{\pi}{2}$. Answer: C



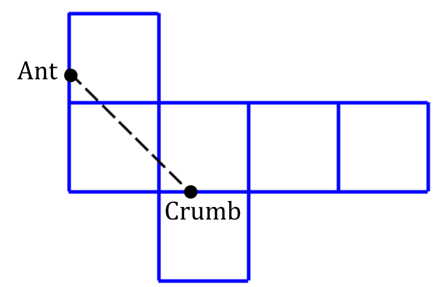
12. $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)\ \Rightarrow\ \cos(A-B)=\cos(A)\cos(-B)-\sin(A)\sin(-B)$
$\Rightarrow\ \cos(A-B)=\cos(A)\cos(B)+\sin(A)\sin(B)$, since cosine is an even function, and sine is odd. Now let $A=60°$ and $B=10°$, so we get: $\cos(60°+10°)=\cos(60°)\cos(10°)-\sin(60°)\sin(10°)$ and $\cos(60°-10°)=\cos(60°)\cos(10°)+\sin(60°)\sin(10°)$. Adding these two results gives: $\cos(70°)+\cos(50°)=2\cos(60°)\cos(10°)=2\left(\frac{1}{2}\right)\cos(10°)=\cos(10°)$. Thus, $x=10°$. Answer: B

13. Dividing by 4^x gives: $\dfrac{9^x - 6^x}{4^x} = \dfrac{2 \cdot 4^x}{4^x} \Rightarrow \left(\frac{9}{4}\right)^x - \left(\frac{6}{4}\right)^x = 2$ or $\left(\frac{3}{2}\right)^{2x} - \left(\frac{3}{2}\right)^x = 2$. Now let $y = \left(\frac{3}{2}\right)^x$, giving: $y^2 - y = 2$ or $(y-2)(y+1) = 0$. So, $y = 2$ or $y = -1$. But, $y = \left(\frac{3}{2}\right)^x$ which is positive, thus $y = 2$. Solving for x: $\left(\frac{3}{2}\right)^x = 2 \Rightarrow x = \dfrac{\ln(2)}{\ln\left(\frac{3}{2}\right)} = \dfrac{\ln(2)}{\ln(3) - \ln(2)}$.

Answer: A

14. Unfolding the cube as shown, reveals a shorter distance than the obvious walk directly down the wall and across the floor (giving a distance of 6). Thus, the shortest path is the hypotenuse of the right triangle with legs of length 4 (i.e. 3+1). Making the shortest distance $4\sqrt{2}$ (we know this is less than 6 since $\sqrt{2} < 1.5$).

Answer: $4\sqrt{2}$



15. Using $2^{10} = 1024$, we can look for a pattern. $\left(2^{10}\right)^2 = 1024^2 = ...76$, $\left(2^{10}\right)^3 = ...76 \cdot 1024 = ...24$, and so we see a pattern after only 2 iterations. We see that positive odd powers of 1024 end with 24, and positive even powers of 1024 end with 76. Hence, $2^{74,207,281} - 1 = 2 \cdot 2^{74,207,280} - 1$ $= 2 \cdot 2^{10 \cdot 7,420,728} - 1 = 2 \cdot 1024^{7,420,728} - 1 = 2 \cdot (...76) - 1 = (...52) - 1 = ...51$.

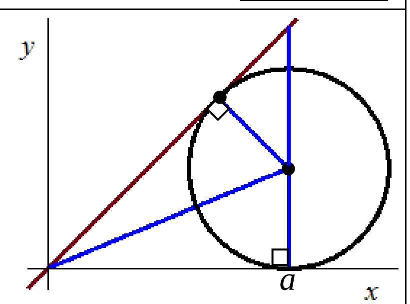
Answer: 51

16. Taking the square root of both sides gives: $\sqrt{x^2 e^x} = \sqrt{4} \Rightarrow x e^{x/2} = 2$, for $x > 0$. Now divide both sides by 2 to get $\frac{x}{2} e^{x/2} = 1$, which now fits the form for the Lambert W function. Thus, $\frac{x}{2} = W(1)$ or $x = 2W(1)$.

Answer: C

17. Let $x = a$ be the x-value we seek, with $a > 0$. Drawing a line from $x = a$ on the x-axis through the center of the circle to the line $y = x$ forms two 45°-45°-90° triangles; a large one whose legs have length a, and a small one with legs of length 1 (the radius of the circle). The small triangle has hypotenuse of length $\sqrt{2}$, making the legs (a) of the larger triangle $\sqrt{2} + 1$.

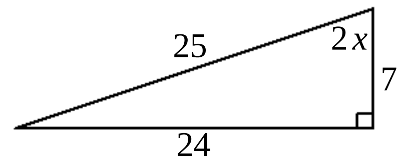
Answer: $\sqrt{2} + 1$



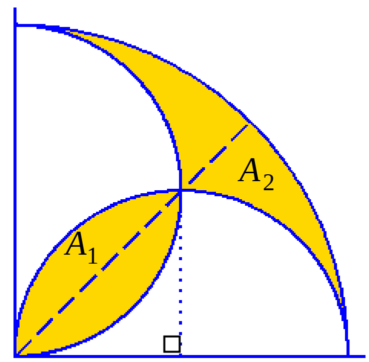
<u>Alternate Solution</u>: Simple right triangle trigonometry, on the triangle whose hypotenuse is from the origin to the center of the circle, tells us $\tan(22.5°) = \frac{1}{a} \Rightarrow a = \frac{1}{\tan(22.5°)}$. Using the half-angle formula for tangent, $\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{\sin(\theta)}$, we get $\tan(22.5°) = \frac{1 - \cos(45°)}{\sin(45°)} = \frac{1 - \sqrt{2}/2}{\sqrt{2}/2} = \sqrt{2} - 1$. So, we get (again), $a = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$.

18. Squaring both sides gives: $\left[\cos(x)+\sin(x)\right]^2=\left(\frac{7}{5}\right)^2 \Rightarrow \cos^2(x)+2\cos(x)\sin(x)+\sin^2(x)=\frac{49}{25}$, which reduces to: $1+2\cos(x)\sin(x)=\frac{49}{25}$ and then: $\sin(2x)=\frac{24}{25}$, since $\sin(2x)=2\cos(x)\sin(x)$. Now form a right triangle with this angle $2x$, which yields $\tan(2x)=\frac{24}{7}$.

Answer: C



19. Let's determine the shaded area in the first quadrant, then we'll multiply by 4 to obtain the entire area. A_1 is the area of the segment cut off by the line $y=x$ (dashed), while A_2 is the area of the region within the sector (bound by the circle of radius 1, $y=x$, and the x-axis) but outside the lower semi-circle of radius ½ .



$$A_1=\begin{pmatrix}\text{area of the ¼ - circle}\\ \text{of radius ½}\end{pmatrix}-\begin{pmatrix}\text{area of the right triangle with}\\ \text{a base and height of ½}\end{pmatrix}$$

$$=\left(\tfrac{1}{4}\cdot\pi\cdot\left(\tfrac{1}{2}\right)^2\right)-\left(\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\right)=\frac{\pi}{16}-\frac{1}{8}$$

Notice the area of the each small semi-circle is $\tfrac{1}{2}\cdot\pi\cdot\left(\tfrac{1}{2}\right)^2=\frac{\pi}{8}$, and the area of the sector in question is $\tfrac{1}{8}\cdot\pi\cdot 1^2=\frac{\pi}{8}$. Since the two areas are equal, $A_1=A_2$. Thus, the area of the shaded region in the first quadrant is $2A_1+2A_2=4A_1=4\left(\frac{\pi}{16}-\frac{1}{8}\right)=\frac{\pi}{4}-\frac{1}{2}$. Therefore, the total shaded area is $4\left(\frac{\pi}{4}-\frac{1}{2}\right)=\pi-2$.

Answer: $\pi-2$

20. Five cards *must* be turned over, which are: A, E, 2, 0, and 6.
(This is a version of the *Wason Selection Task*.)

Answer: C