New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2017

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. The Dirichlet function, $D(x)$, is defined for all real numbers as $D(x)=\begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$.

   Which of the following statements are/is true?

   I. $D(-\pi)=0$ II. $D\left(\frac{22}{7}-\pi\right)=1$ III. $D\left(\pi^2\right)=1$ IV. D is invertible

   A) I only B) II and III only C) I, II, and III only D) I, II, III, and IV

2. What is the least common multiple of $\frac{5}{6}$, $\frac{10}{9}$, and $\frac{8}{15}$? (i.e. What is the smallest positive number that when divided by each, leaves a zero remainder?)

3. My new car gets 60% percent better fuel efficiency, measured in miles per gallon, than my old car. However, the new car requires I use super unleaded gasoline, which is 20% more expensive per gallon than the regular unleaded gasoline that I use in my old car. By what percent will I save money by using my new car instead of my old car for a long trip?

   A) 12% B) 25% C) $33\frac{1}{3}$% D) 40%

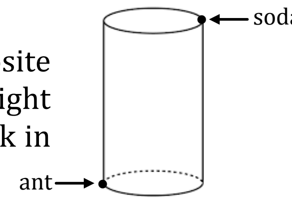
4. If m and n are positive integers such that neither one contains a zero and $mn=100000$, then what is $m+n$? Note: The solution is unique.

5. A math textbook has n pages, consecutively numbered 1 through n. If a total of 1737 digits were used to number the pages, then what is n?

6. What is the sum of all real x-values that satisfy $x^{\sqrt{x}}=x\sqrt{x}$?

7. How many zeros (i.e. solutions to $f(x)=0$) does $f(x)=\sin(\ln(x))$ have on the interval $x\in(0,1)$?

A) none B) exactly 1 C) exactly 2 D) infinitely many



8. An ant is at the bottom of a soda can when it detects a drop of soda on the opposite side of the top of the can, as shown. Assuming the soda can is a cylinder of height 12 cm and diameter 6 cm, what is the shortest distance (in cm) the ant can walk in order to get the drop of soda?

A) $6\sqrt{5}$ B) $3\sqrt{16+\pi^2}$ C) 18 D) $12+3\pi$

9. If $\log_2\left(\log_3\left(\log_5\left(\log_7(n)\right)\right)\right)=2017$, then n has how many different prime factors?

A) 1 B) 4 C) 5 D) more than 5

10. In a 1 mile race, Alvin beat Brianna by 0.1 of a mile and Brianna beat Charles by 0.2 of a mile. Assuming they ran at a constant rate throughout, by how much (of a mile) did Alvin beat Charles?

11. If $2^x=3^y=6$, then which one of the following equations is correct?

A) $xy=x-y$ B) $xy=x+y$ C) $xy=\frac{x}{y}+\frac{y}{x}$ D) $xy=6^{2/3}$

12. For $0<x<\frac{\pi}{4}$, the size order of the functions: $\tan(x)^{\tan(x)}$, $\tan(x)^{\cot(x)}$, $\cot(x)^{\tan(x)}$, and $\cot(x)^{\cot(x)}$ is:

A) $\tan(x)^{\tan(x)}<\tan(x)^{\cot(x)}<\cot(x)^{\cot(x)}<\cot(x)^{\tan(x)}$

B) $\tan(x)^{\cot(x)}<\tan(x)^{\tan(x)}<\cot(x)^{\tan(x)}<\cot(x)^{\cot(x)}$

C) $\tan(x)^{\cot(x)}<\cot(x)^{\tan(x)}<\tan(x)^{\tan(x)}<\cot(x)^{\cot(x)}$

D) NOTA

13. The three vertices of a triangle are: $(0,0)$, $(\cos(\theta),\sin(\theta))$, and $(\sin(\theta),\cos(\theta))$. For how many values of $\theta\in[0,\pi]$ will the area of the triangle be $\frac{1}{2}$?

A) exactly 1 B) exactly 2 C) exactly 3 D) more than 3

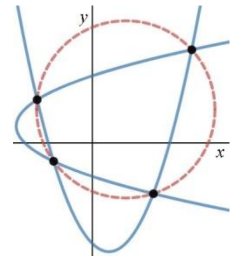
14. How many real values for x solve the equation $|x|+\frac{4|x|}{x}=\frac{4}{x}$?

A) none B) exactly one C) exactly two D) more than two

15. $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, which equals $3{,}628{,}800$. If one of the factors is randomly selected, what is the probability it will be an odd number? Remember to include 1 and $10!$ as factors!

16. What is the value of $\displaystyle\sum_{n=2}^{2017} \frac{1}{\log_n(2017!)} = \frac{1}{\log_2(2017!)} + \frac{1}{\log_3(2017!)} + \cdots + \frac{1}{\log_{2017}(2017!)}$?

A) between 0 and 1 B) 1 C) between 1 and 2 D) 2 or greater



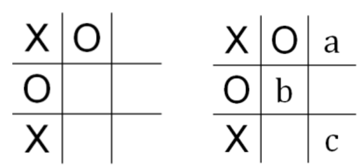
17. The two parabolas given by $y = x^2 - x - 3$ and $x = y^2 - y - 2$ intersect at four points which lie on a circle, as shown. What is the radius of the circle?

18. Let $P_3(x)$ be a 3rd degree polynomial with integer coefficients. If $P_3(2017) = 2017$, then $P_3(1)$ could *not* equal which of the following values?

A) -2017 B) 0 C) 2017 D) NOTA

19. What is the area of a trapezoid with sides of length 1, 2, 3, and 4?

20. Suppose you are playing a game of tic-tac-toe, with you as "X" and your opponent as "O." The first diagram shows the game after four moves, and it is now your turn. Using the second diagram, where should you place an "X" in order to guarantee that you win? Note: You win by getting three X's in a line before your opponent gets three O's in a line.



A) b only
B) a or b only
C) b or c only
D) a, b, or c

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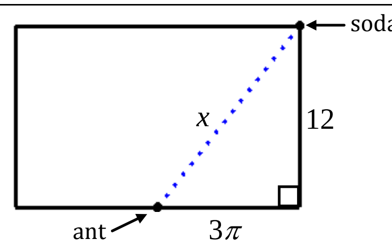
New York State Mathematics Association of Two-Year Colleges

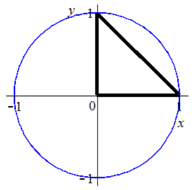
Math League Contest ~ Fall 2017 ~ Solutions

1.	π is irrational, which makes $-\pi$ irrational, thus $D(-\pi)=0$. The sum or difference of an irrational and rational number is irrational, thus $D\left(\frac{22}{7}-\pi\right)=0$. π is not the square root of a rational number, which makes π^2 irrational, making $D\left(\pi^2\right)=0$. Since multiple values make $D(x)=0$, i.e. any irrational number, D cannot be invertible. Hence, only statement I is true. Answer: A
2.	Writing each fraction in prime factored form gives: $\frac{5}{6}=2^{-1}\cdot 3^{-1}\cdot 5^1$, $\frac{10}{9}=2^1\cdot 3^{-2}\cdot 5^1$, and $\frac{8}{15}=2^3\cdot 3^{-1}\cdot 5^{-1}$, while noting that the LCM is the product of each prime factor with the highest power. Thus, the LCM $=2^3 3^{-1} 5^1=\frac{40}{3}$. Answer: $\frac{40}{3}$ Alternate Solution: Let N be the LCM. Thus, ① $N=\frac{5}{6}a$, ② $N=\frac{10}{9}b$, and ③ $N=\frac{8}{15}c$, with a, b, and c being positive integers. Solving ① and ② by eliminating N gives: $\frac{5}{6}a=\frac{10}{9}b \Rightarrow a=\frac{4}{3}b$. Doing likewise for ② and ③ gives: $\frac{10}{9}b=\frac{8}{15}c \Rightarrow b=\frac{12}{25}c$. However, these must be positive integers, making 25 the smallest value of c (thus $b=12$ and $a=16$). Hence, $N=\frac{8}{15}\cdot 25=\frac{40}{3}$.
3.	Let d be the distance travelled (in miles), and e be the fuel economy of my old car (in miles/gallon). Thus, the number of gallons of gasoline needed by my old car for the trip is d/e. The new car gets 60% better fuel economy, which converts to $1.6e=\frac{8}{5}e$. Therefore, the number of gallons needed by my new car for the trip is $d/\left(\frac{8}{5}e\right)=\frac{5}{8}(d/e)$ gallons, i.e. $\frac{5}{8}$ of what my old car would need. But, the gas for my new car costs 20% more, making the cost 1.2 or $\frac{6}{5}$ times the cost of the old car. Hence, it will cost $\frac{5}{8}\cdot\frac{6}{5}=\frac{3}{4}$ (or 75%) the cost for the new car as compared to my old car for gas. Consequently, saving me 25%. Answer: B
4.	$100000=10^5=(2\cdot 5)^5=2^5\cdot 5^5=32\cdot 3125$. Thus, m and n must be 32 and 3125, in either order, making the sum 3157. Answer: 3157
5.	Pages 1 through 9 requires 9 digits, and pages 10 through 99 requires $2\cdot 90=180$ digits, giving a partial total of 189 digits. This leaves $1737-189=1548$ digits for which to account. Pages 100 through 999 each use 3 digits. $1548/3=516$. Thus, there are 516 pages that follow page 99, making $99+516=615$ pages. Answer: 615

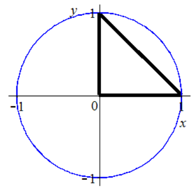
6.	$x^{\sqrt{x}} = x\sqrt{x}$ can be written as $x^{x^{1/2}} = x^{3/2}$, or $x^{x^{1/2}} - x^{3/2} = 0$. The left-hand-side factors to give $x^{3/2}\left(x^{x^{1/2}-3/2}-1\right)=0$, yielding ① $x^{3/2}=0$ and ② $x^{x^{1/2}-3/2}-1=0$. Equation ① gives $x=0$ (which may be excluded since it gives 0^0, or may be included since it does not alter the sum), and ② gives either $x=1$ or $x^{1/2}-3/2=0$, with $x \neq 0$. The latter equation gives us $x=9/4$. Thus, the sum of the real solutions is $1+9/4=13/4$. Answer: 13/4 or 3.25
7.	The function $\ln(x)$ approaches $-\infty$ as x approaches 0 from the right. Thus, $\sin(\ln(x))$ oscillates (between -1 and 1) infinitely many times as x approaches 0 from the right. Therefore, the function $f(x)=\sin(\ln(x))$ crosses the x-axis infinitely many times. Answer: D
8.	The shortest distance between two points is along the line connecting them. Thus, if we cut open the cylindrical wall of the can at the drop of soda we see the shortest distance is the line from the middle of the base of the wall to the drop of soda, the dotted line. The length of the wall is the circumference of the can, i.e. 6π cm. The Pythagorean theorem gives us the desired distance, x:  $(3\pi)^2+12^2=x^2 \Rightarrow x=\sqrt{144+9\pi^2}=3\sqrt{16+\pi^2}$ Answer: B
9.	$\log_2\left(\log_3\left(\log_5\left(\log_7(n)\right)\right)\right)=2017 \Rightarrow \log_3\left(\log_5\left(\log_7(n)\right)\right)=2^{2017} \Rightarrow \log_5\left(\log_7(n)\right)=3^{2^{2017}}$ $\Rightarrow \log_7(n)=5^{3^{2^{2017}}} \Rightarrow n=7^{5^{3^{2^{2017}}}}$ Thus, n equals 7 raised to a *very large* positive integer power. Therefore, 7 is the only prime factor of n. Answer: A
10.	Let t_a be the time it takes Alvin to run a mile, t_b be the time it takes Brianna to run a mile, S_b be Brianna's speed, and S_c be Charles' speed. Thus, ① $S_b \cdot t_b = 1$ and ② $S_b \cdot t_a = 0.9$ (since Brianna lost to Alvin by 0.1 mile). Solving ① and ② for t_b gives $t_b = t_a/0.9$. Also, we get ③ $S_c \cdot t_b = 0.8$ (since Charles lost to Brianna by 0.2 mile). Substituting $t_b = t_a/0.9$ into ③ gives $S_c \cdot t_a/0.9 = 0.8$ or $S_c \cdot t_a = 0.72$, i.e. by the time Alvin has run the mile, Charles has run 0.72 mile. Hence, Alvin beat Charles by 0.28 mile. Answer: 0.28
11.	$2^x = 3^y = 6 \Rightarrow 2^x = 6$ and $3^y = 6 \Rightarrow (2^x)^y = 6^y$ and $(3^y)^x = 6^x \Rightarrow 2^{xy} = 6^y$ and $3^{xy} = 6^x$ Thus, $2^{xy} \cdot 3^{xy} = 6^y \cdot 6^x \Rightarrow 6^{xy} = 6^{x+y} \Rightarrow xy = x+y$. Answer: B
12.	For $0 < x < \pi/4$: $\tan(x) \in (0,1)$, with $\tan(0)=0$ and $\tan(\pi/4)=1$; and $\cot(x) \in (1,\infty)$, with $\lim_{x\to 0^+} \cot(x) \to +\infty$ and $\cot(\pi/4)=1$. Now let's consider the behavior of each function near the end-points and in between: $\lim_{x\to 0^+} \tan(x)^{\tan(x)} = 1$, $\lim_{x\to \pi/4} \tan(x)^{\tan(x)} = 1$, with $0 < \tan(x)^{\tan(x)} < 1$ in between; $\lim_{x\to 0^+} \tan(x)^{\cot(x)} = 0$, $\lim_{x\to \pi/4} \tan(x)^{\cot(x)} = 1$, with $0 < \tan(x)^{\cot(x)} < \tan(x)^{\tan(x)}$ in between; $\lim_{x\to 0^+} \cot(x)^{\tan(x)} = 1$, $\lim_{x\to \pi/4} \cot(x)^{\tan(x)} = 1$, with $1 < \cot(x)^{\tan(x)} < \cot(x)^{\cot(x)}$ in between; $\lim_{x\to 0^+} \cot(x)^{\cot(x)} \to +\infty$, $\lim_{x\to \pi/4} \cot(x)^{\cot(x)} = 1$, with $\cot(x)^{\cot(x)} > \cot(x)^{\tan(x)}$ in between. Thus, $\tan(x)^{\cot(x)} < \tan(x)^{\tan(x)} < \cot(x)^{\tan(x)} < \cot(x)^{\cot(x)}$. Answer: B

13. All triangles formed have one vertex at the origin while the other two vertices lie on a unit circle centered at the origin. Since the base of all such triangles have a length of 1, we seek the angles that give a height of 1 as well. There are only three angles, θ, that give the desired result: $\theta = 0$, $\frac{\pi}{2}$, and π, as shown in the diagrams. **Answer: C**

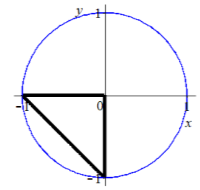
$\theta = 0$:



$\theta = \frac{\pi}{2}$:



$\theta = \pi$:



14. Since $|x| \equiv \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$, we consider two cases.

Case I: $x<0 \Rightarrow -x + \frac{4(-x)}{x} = \frac{4}{x} \Rightarrow -x-4 = \frac{4}{x} \Rightarrow x^2+4x+4=0 \Rightarrow (x+2)^2 = 0 \Rightarrow x=-2$

Case II: $x \geq 0 \Rightarrow x + \frac{4x}{x} = \frac{4}{x} \Rightarrow x+4 = \frac{4}{x} \Rightarrow x^2+4x-4=0 \Rightarrow x = -2 \pm 2\sqrt{2}$, two real solutions, but only $-2+2\sqrt{2} \geq 0$. Hence, there are exactly two real solutions. **Answer: C**

15. $10! = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = (2\cdot 5)\cdot 3^2\cdot 2^3\cdot 7\cdot(2\cdot 3)\cdot 5\cdot 2^2\cdot 3\cdot 2 = 2^8\cdot 3^4\cdot 5^2\cdot 7^1$ Thus, factors of 10! have the form $2^a\cdot 3^b\cdot 5^c\cdot 7^d$, where $a\in\{0,1,2,3,4,5,6,7,8\}$, $b\in\{0,1,2,3,4\}$, $c\in\{0,1,2\}$, and $d\in\{0,1\}$. Hence, 10! has $9\cdot 5\cdot 3\cdot 2 = 270$ factors. The odd factors are of the form $3^b\cdot 5^c\cdot 7^d$, i.e. no even factor, giving $5\cdot 3\cdot 2 = 30$ odd factors. Therefore, the probability of an odd factor being selected at random is $\frac{30}{270} = \frac{1}{9}$. **Answer: $\frac{1}{9}$**

16. $\log_n(2017!) = \frac{\ln(2017!)}{\ln(n)} \Rightarrow \frac{1}{\log_n(2017!)} = \frac{\ln(n)}{\ln(2017!)} \Rightarrow \sum_{n=2}^{2017} \frac{1}{\log_n(2017!)} = \sum_{n=2}^{2017} \frac{\ln(n)}{\ln(2017!)}$

Thus, $\sum_{n=2}^{2017} \frac{1}{\log_n(2017!)} = \frac{\ln(2)}{\ln(2017!)} + \frac{\ln(3)}{\ln(2017!)} + \frac{\ln(4)}{\ln(2017!)} + \cdots + \frac{\ln(2017)}{\ln(2017!)}$

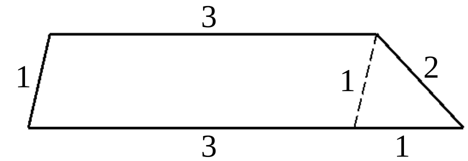
$= \frac{\ln(2)+\ln(3)+\ln(4)+\cdots+\ln(2017)}{\ln(2017!)} = \frac{\ln(2\cdot 3\cdot 4\cdot\ldots\cdot 2017)}{\ln(2017!)} = \frac{\ln(2017!)}{\ln(2017!)} = 1$ **Answer: B**

17. Since the four points of intersection satisfy both equations, they must also satisfy the sum of the two equations. Thus, $y+x = x^2-x-3+y^2-y-2 \Rightarrow 5 = x^2-2x+y^2-2y$. Completing the square on the x-terms and y-terms yields: $5+1+1 = (x-1)^2+(y-1)^2 \Rightarrow (x-1)^2+(y-1)^2 = \sqrt{7}^2$. Hence, the four points of intersection lie on a circle of radius $\sqrt{7}$, centered at (1,1). **Answer: $\sqrt{7}$**

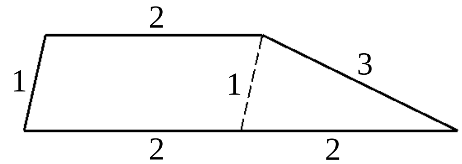
18. Let $P_3(x) = ax^3+bx^2+cx+d$, where a,b,c, and d are integers. $P_3(2017) = 2017$ tells us ① $2017^3a+2017^2b+2017c+d = 2017$. Now letting $P_3(1) = k$, gives ② $a+b+c+d = k$. Subtracting ② from ① yields: ③ $(2017^3-1)a+(2017^2-1)b+2016c = 2017-k$. Equation ③ has all even coefficients on the left-hand-side (LHS). Thus, for all integers a,b,c, and d the LHS will be an even number. Therefore, the RHS must also be even, i.e. $2017-k$ must be even. Hence, among the choices, only $k=0$ would not make $2017-k$ even. **Answer: B**

19. First we must determine the configuration of the quadrilateral with sides 1, 2, 3, and 4 that yields a trapezoid (i.e. a quadrilateral with a pair of parallel sides). Let's consider the possibilities, keeping the side of length 4 as a base. Note: Only the correct case will be drawn to scale.

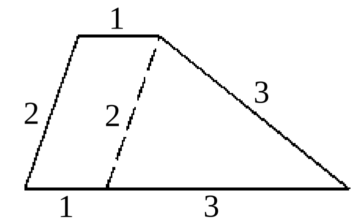
Case I: Bases of length 3 and 4, giving sides of length 1 and 2. Sliding the side of length 1 over (the dashed line) to form a "triangle" with sides 1, 1, and 2 — which is impossible since a triangle must have the sum of the two shorter sides be greater than the longest side.



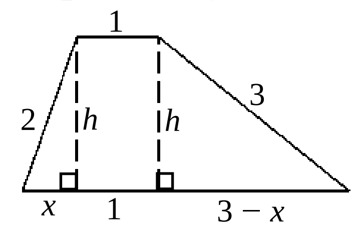
Case II: Bases of length 2 and 4, giving sides of length 1 and 3. Sliding the side of length 1 over (the dashed line) to form a "triangle" with sides 1, 2, and 3 — which is also impossible.



Case III: Bases of length 1 and 4, giving sides of length 2 and 3. Sliding the side of length 2 over (the dashed line) to form a triangle with sides 2, 2, and 3 — which *is* possible!



Now we only need to determine the height, h, to obtain the area. Drawing the two heights shown yields two right triangles. Labeling the base of the smaller triangle by x gives the relationships shown.



Applying the Pythagorean theorem to the smaller triangle gives: $x^2+h^2=2^2 \Rightarrow h=\sqrt{4-x^2}$.

Now applying the Pythagorean theorem to the larger triangle, we get $(3-x)^2+h^2=3^2 \Rightarrow h=\sqrt{9-(3-x)^2}$ or $h=\sqrt{6x-x^2}$. Equating the two results: $\sqrt{4-x^2}=\sqrt{6x-x^2} \Rightarrow x=\frac{2}{3}$ Thus, $h=\sqrt{4-\left(\frac{2}{3}\right)^2}=\sqrt{\frac{32}{9}}=\frac{4}{3}\sqrt{2}$. Therefore, the area is $\frac{1}{2}(1+4)\cdot\frac{4}{3}\sqrt{2}=\frac{10}{3}\sqrt{2}$.

Answer: $\frac{10}{3}\sqrt{2}$

20. Only *b* or *c* will assure your victory.

Answer: C

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