New York State Mathematics Association of Two-Year Colleges

NYSMATYC 1967

Math League Contest ~ Fall 2018

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, π, etc. NOTE: NOTA = None Of These Answers.

1. Function f satisfies the relation $f(xy) = \frac{f(x)}{y}$, for $x, y \in (0,\infty)$. If $f(\pi) = e$, then what is the value of $f\left(\frac{1}{\pi}\right)$?

   A) $\frac{e}{\pi^2}$ B) $\frac{1}{e}$ C) $\frac{e}{\pi}$ D) $\frac{\pi}{e}$ E) $e\pi^2$

2. The absolute value of both roots of the quadratic equation $x^2 + 81x + k = 0$ are prime numbers. How many values of k are possible?

   A) none B) exactly one C) exactly two D) exactly three E) four or more

3. When it is 12 noon in New York City it is 5 PM on the same day in London and 3 AM the next day in Melbourne, Australia. A traveler takes a flight from London at 3 PM on a Thursday and lands in New York City 8 hours later. She stays in New York for a brief time, taking a flight to Melbourne only 14 hours after landing. If the flight from New York to Melbourne is 21 hours, then what *day and time* is it when she lands in Melbourne?

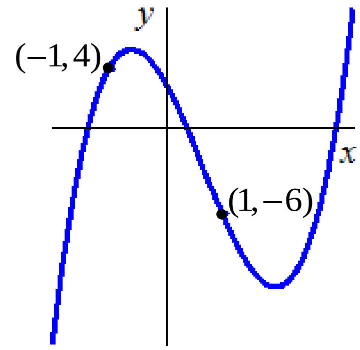
4. A car with five tires (four road tires and a spare) traveled 40,000 miles. All five tires were equally used so they each had the same wear. How many miles was each tire used?

   A) 8,000 B) 10,000 C) 30,000 D) 32,000 E) 36,000

5. Two fair 6-sided dice are rolled, and it is revealed that one of the dice rolled a 5. What is the probability that the other die rolled a 3?

   A) $\frac{1}{18}$ B) $\frac{1}{12}$ C) $\frac{1}{6}$ D) $\frac{2}{11}$ E) $\frac{2}{9}$

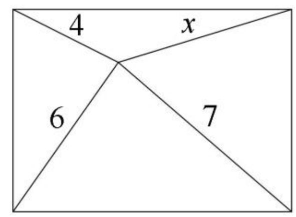
6. Part of the graph of $f(x)=ax^3+bx^2+cx+d$ is shown, containing the points indicated. What is the value of $a+c$?
A) -5 B) -2 C) -1 D) 2
E) There is not enough information.



7. What is the area enclosed by the graph of $|2x|+|3y|=6$?

8. For the first *n* years of my life my mother would make a cake for my birthday, and for the first third of those years she would put the number of candles on the cake equal to my age. For the last two-thirds of those years, when I got too old for so many candles, she put only one candle on the cake. I have seen a total of 150 candles from all this. What is *n* (i.e. at what age did this process end)?

9. Four line segments are drawn from the corners of a rectangle, with the lengths shown, that meet at a single point. What is the measure of the 4th segment, *x*? Note: The diagram is not drawn to scale.

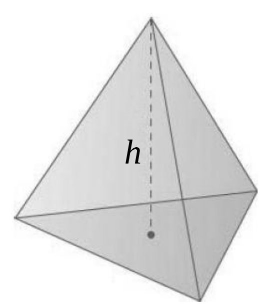


10. What is the smallest positive integer *n* that makes $\frac{n}{1776}$ a terminating decimal?
For example, $1/2=0.5$ is a terminating decimal, while $1/3=0.\overline{3}$ is not.

11. The graphs $y=\left(\log_2(x)-2\right)^{x^2-2}$ and $y=1$ intersect at how many points?
A) none B) exactly one C) exactly two D) exactly three E) four or more

12. Container A has 1 liter of pure alcohol, while Container B has 2 liters of pure water. I take a teaspoon of the pure alcohol and put into the water. I thoroughly stir the alcohol-water mixture, then take a teaspoon of that mixture and put it into the alcohol.
A) There is now twice as much water in Container A than alcohol in Container B.
B) There is now twice as much alcohol in Container B than water in Container A.
C) The amount of water in Container A is equal to the amount of alcohol in Container B.
D) There is no way to be certain, since the units (liters and teaspoons) are inconsistent.
E) NOTA

13. A regular tetrahedron is a solid bound by four equilateral triangular faces, as shown. If each edge has a length of 1, then what is the height, *h*?
A) $\frac{\sqrt{2}}{2}$ B) $\frac{3}{4}$ C) $\frac{4}{5}$ D) $\frac{\sqrt{6}}{3}$ E) $\frac{\sqrt{3}}{2}$

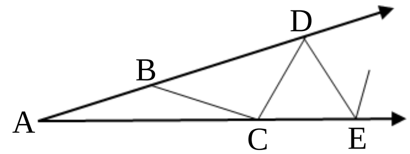


14. Which of the following functions is/are odd (i.e. whose graph is symmetric about the origin)?

I. $y=\tan(\cos(x))$ II. $y=\tan(\sin(x))$ III. $y=\ln\left(\sqrt{x^2+1}+x\right)$

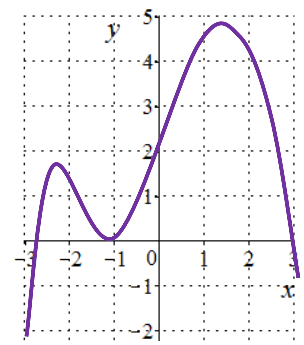
A) I only B) II only C) I and II only D) I and III only E) II and III only

15. A sequence of isosceles triangles is constructed with $AB=BC$, then $BC=CD$, followed by $CD=DE$, and so on, as shown in the diagram (not drawn to scale). If $\measuredangle BAC=15°$, then how many such triangles can be drawn?



A) exactly three B) exactly four C) exactly five D) exactly six E) more than six

16. The graph of the function $y=f(x)$ is shown. How many solutions does the equation $f\left(f(x)\right)=4$ have?



A) exactly two B) exactly four C) exactly six
D) more than six E) NOTA

17. What is the area enclosed by the graph of $\log_{10}\left(10+x^2+y^2\right)\le 1+\log_{10}(1+x-2y)$?

18. How many distinct solutions are there for the equation: $\pm x\pm x\pm\cdots\pm x=2018$, where there are 2018 x's. For example with 3 x's, the equation $\pm x\pm x\pm x=2018$ refers to the eight possible equations: $+x+x+x=2018$, $+x+x-x=2018$, $+x-x+x=2018$, $-x+x+x=2018$, $+x-x-x=2018$, $-x+x-x=2018$, $-x-x+x=2018$, and $-x-x-x=2018$, which have the following solutions (respectively): $\frac{2018}{3}$, 2018, 2018, 2018, –2018, –2018, –2018, and $-\frac{2018}{3}$, giving only 4 distinct solutions.

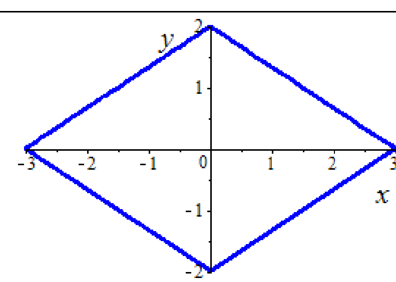
19. While using my calculator to find the sine of an angle, x, I neglected to change the angle mode from degrees to radians. Thus, the calculator gave me $\sin(x°)$ when I meant for x to be in radian measure. Fortunately, the answers were the same! What is the smallest positive value for x where this is true?

20. A small company uses a 5-digit employee code that must adhere to the following rules: ① only the digits 0, 1, 2, 3, and 4 can be used, and each must be used, ② The second digit has a value that is twice the first digit, and ③ the value of the third digit is less than the value of the fifth digit. If the last digit of an acceptable code is 1, then which of the following must be true?

A) The first digit is 2. B) The second digit is 0. C) The third digit is 3.
D) The fourth digit is 4. E) The fourth digit is 0.

New York State Mathematics Association of Two-Year Colleges

Math League Contest ~ Fall 2018 ~ Solutions

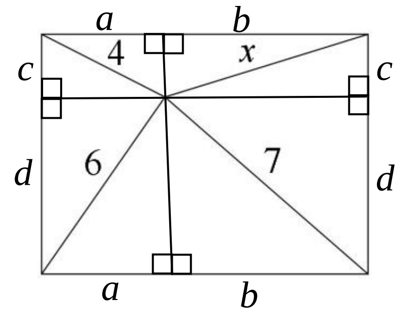
1.	$f\left(\frac{1}{\pi}\right)=f\left(\pi\cdot\frac{1}{\pi^2}\right)=\dfrac{f(\pi)}{1/\pi^2}=e\cdot\pi^2.$ **Answer: E**
2.	If the roots are r_1 and r_2, then $(x-r_1)(x-r_2)=x^2+81x+k$. Thus, $r_1\cdot r_2=k$ and $-(r_1+r_2)=81$. In order for the sum, or difference, of two integers to be odd, one must be even and one must be odd. The only even prime number is 2. Therefore, one of the roots is 2 or -2. If $r_1=2$, then $-(2+r_2)=81$ and $r_2=-83$ (with 83 being prime), making $k=2\cdot(-83)=-166$. If $r_1=-2$, then $-(-2+r_2)=81$ and $r_2=-79$ (with 79 being prime), making $k=-2\cdot(-79)=158$. Hence, there are two different values for k that are possible. **Answer: C**
3.	Let's just keep track of the day and time in Melbourne. The traveler begins her journey in London on a Thursday at 3 PM, which is Friday at 1 AM in Melbourne (10 hours later). She lands in NY 8 hours later, making it Friday at 9 AM in Melbourne. Her next flight from NY to Melbourne is 14 hours later, thus leaving NY when it is Friday at 11 PM in Melbourne. The flight takes 21 hours, so she lands in Melbourne on Saturday at 8 PM. **Answer: Saturday, 8 PM**
4.	Each tire is utilized $4/5$ of the time. Thus, each tire is driven $4/5$ of 40,000 miles or 32,000 miles. **Answer: D**
5.	There are $6\cdot 6$ or 36 equally likely outcomes, with 11 of them having a 5: (5, 1), (1, 5), (5, 2), (2, 5), (5, 3), (3, 5), (5, 4), (4, 5), (5, 6), (6, 5), or (5, 5). Among these 11 equally likely outcomes, given a 5 was rolled, only 2 of them have the other roll being a 3. Hence, the probability is $\frac{2}{11}$. **Answer: D**
6.	$f(1)=a+b+c+d$ and $f(-1)=-a+b-c+d$, with $(a+b+c+d)-(-a+b-c+d)=2a+2c$. Thus, $a+c=\frac{1}{2}\left(f(1)-f(-1)\right)=\frac{1}{2}(-6-4)=-5$. **Answer: A**
7.	The absolute values in the equation, $\lvert 2x\rvert+\lvert 3y\rvert=6$, makes the graph symmetric about the *x*-axis *and* the *y*-axis. Thus, we need only obtain the graph in the first quadrant then do the appropriate reflections. In the first quadrant, where *x* and *y* are both positive, we can dispense with the absolute values to obtain $2x+3y=6$, whose graph is the line segment from $(0,2)$ to $(3,0)$. Then doing the proper reflections by symmetry, gives the rhombus shown.  The rhombus can be broken up into four right triangles each with a base of length 3 and height 2. Hence, the area is $4\cdot\frac{1}{2}\cdot 3\cdot 2=12$. **Answer: 12**
8.	Since n must be a multiple of 3, let $n=3m$. Thus, for the first m years, I saw $1+2+3+\ldots+m$ candles, which equals $\frac{1}{2}m(m+1)$ candles. Then, for the next $2m$ years, I saw $2m$ candles. This makes the total number of candles $\frac{1}{2}m(m+1)+2m=\frac{1}{2}m^2+\frac{5}{2}m$. If the total is 207 then $\frac{1}{2}m^2+\frac{5}{2}m=150$. Solving for m gives: $\frac{1}{2}m^2+\frac{5}{2}m-150=0 \Rightarrow m^2+5m-300=0 \Rightarrow (m+20)(m-15)=0 \Rightarrow m=-20$ or $m=15$. Taking the positive answer then gives $n=3\cdot 15=45$. **Answer: 45**

9. Drawing the horizontal and vertical lines through the common interior point and labeling the segments as shown gives the relations: ① $a^2 + d^2 = 6^2$, ② $a^2 + c^2 = 4^2$, ③ $b^2 + d^2 = 7^2$, and ④ $b^2 + c^2 = x^2$

Subtracting equation ② from equation ① gives: $d^2 - c^2 = 20$.

Subtracting equation ④ from equation ③ gives: $d^2 - c^2 = 49 - x^2$.

Thus, $49 - x^2 = 20$ or $x = \sqrt{29}$. **Answer: $\sqrt{29}$**



10. A non-whole number fraction terminates only if the denominator factors as $2^m \cdot 5^n$ (i.e. it has only 2's and/or 5's as factors). Since $1776 = 2^4 \cdot 111$, with 111 not having 2 or 5 as a factor, we need the numerator to have 111 as a factor to eliminate it from the denominator. **Answer: 111**

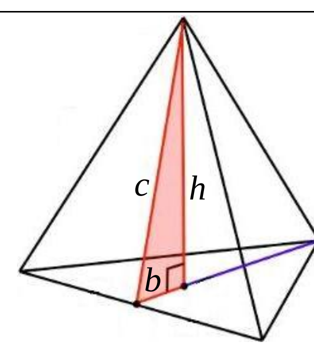
11. We seek solutions to $\left(\log_2(x) - 2\right)^{x^2-2} = 1$ within the domain of both functions, i.e. for $x > 0$. In order for the equation to be true, one of the following cases must be satisfied:

1. $\log_2(x) - 2 = 1$, regardless of what $x^2 - 2$ equals. Thus, $\log_2(x) = 3 \Rightarrow x = 8$.
2. $x^2 - 2 = 0$, with $\log_2(x) - 2 \neq 0$. Thus, $x = \pm\sqrt{2}$. However, $x > 0$, so $x = +\sqrt{2}$.
3. $\log_2(x) - 2 = -1$ with $x^2 - 2$ even. $\log_2(x) - 2 = -1 \Rightarrow x = 2$, which also makes $x^2 - 2$ even.

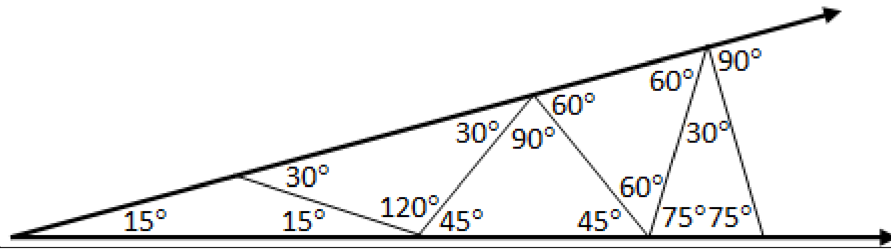
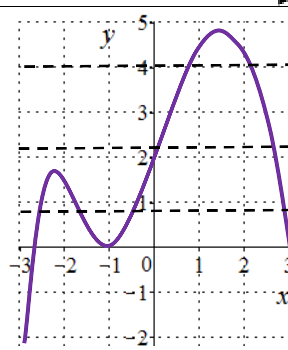
Giving exactly three solutions, and thus three points of intersection. **Answer: D**

12. Instead of using a teaspoon measurement for the transfer of the liquids, let's call it volume v. Thus, a volume v of alcohol was taken from Container A and thoroughly mixed with the 2 liters of water in Container B. Hence, the concentration of alcohol in Container B is now $\frac{v}{2+v}$, and the concentration of water is $\frac{2}{2+v}$. Now an amount v is taken from Container B, leaving it again with a volume of 2 liters. The concentrations of alcohol and water in Container B and the volume v that was removed (the teaspoon) are the same: $\frac{v}{2+v}$ for alcohol and $\frac{2}{2+v}$ for water. Therefore, the amount of alcohol in Container B is $2 \cdot \frac{v}{2+v} = \frac{2v}{2+v}$, and the amount of water in the volume v (which will then be poured into Container A) is $v \cdot \frac{2}{2+v} = \frac{2v}{2+v}$ -- the same! **Answer: C**

13. By symmetry, the segment that represents the height must land at the median of the triangular base, making b equal to ⅓ of the height of the equilateral triangle with edges of length 1 (this can be obtained by drawing the medians on the base triangle to determine b). Thus, $b = \frac{1}{3} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6}$, with c being the height of a triangular face, giving $c = \frac{\sqrt{3}}{2}$. Using the Pythagorean theorem on the shaded triangle to determine h, we get: $\left(\frac{\sqrt{3}}{6}\right)^2 + h^2 = \left(\frac{\sqrt{3}}{2}\right)^2$. Solving for h gives: $h^2 = \frac{3}{4} - \frac{1}{12} \Rightarrow h = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$.



Answer: D

14.	Recall, an odd function satisfies the relation $f(-x)=-f(x)$. $\tan(\cos(-x))=\tan(\cos(x))$, since cosine is an even function, i.e. it satisfies $f(-x)=f(x)$. Hence, "I" is an even function. $\tan(\sin(-x))=\tan(-\sin(x))=-\tan(\sin(x))$, since both sine and tangent or odd functions. Hence, "II" is an odd function. $\ln\left(\sqrt{(-x)^2+1}-x\right)=\ln\left(\sqrt{x^2+1}-x\right)=\ln\left(\frac{\sqrt{x^2+1}-x}{1}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)=\ln\left(\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}\right)=\ln\left(\frac{1}{\sqrt{x^2+1}+x}\right)$ $=-\ln\left(\sqrt{x^2+1}+x\right)$, making "III" an odd function. Answer: E
15.	Drawing the isosceles triangles with the interior angles shows that once get to 90°, we can draw no more triangles. Hence, five is the maximum that can be drawn. Answer: C 
16.	We can see from the graph where the horizontal line $y=4$ intersects $y=f(x)$, that there are two x-values for which $f(x)=4$ ($x\approx 0.8$ and $x\approx 2.1$). Thus, we need the *inner* $f(x)$ to be either of those two x-values. The other two horizontal lines, $y\approx 0.8$ and $y\approx 2.1$, show that there are four x-values ($x\approx -2.6, x\approx -1.6, x\approx -0.5$ and $x\approx 3.9$) that yield $f(x)\approx 0.8$ and two x-values ($x\approx 0.1$ and $x\approx 2.7$) that yield $f(x)\approx 2.1$. Hence, there are 6 x-values such that $f\left(f(x)\right)=4$. Answer: C 
17.	$\log_{10}\left(10+x^2+y^2\right)\le 1+\log_{10}(1+x-2y) \Rightarrow 10^{\log_{10}\left(10+x^2+y^2\right)}\le 10^{1+\log_{10}(1+x-2y)}$ $\Rightarrow 10+x^2+y^2\le 10(1+x-2y) \Rightarrow x^2-10x+y^2+20y\le 0$, completing the square gives: $(x-5)^2+(y+10)^2\le 125$, which represents the area within a circle of radius $\sqrt{125}$ -- centered at $(5,-10)$. Thus, the area enclosed is $\pi\cdot\sqrt{125}^{\,2}=125\pi$. Answer: 125π
18.	We will get duplicate answers whenever the number of pluses and minuses don't change, when they are just a reordered. Thus, we need only count the number of equations that have a different number of pluses and minuses. For 2018 x's, we can have 0 pluses and 2018 minuses, or 1 plus and 2017 minuses, or 2 pluses and 2016 minuses, or . . . , up to 2018 pluses and 0 minuses, for a total of 2019 different equations and solutions. However, since 2018 is even, this includes 1009 pluses and 1009 minuses, which would yield the equation $0=2018$, which is impossible. Hence, there are only 2018 distinct solutions. Answer: 2018
19.	The smallest positive angles will be in Quadrant I when measured in degrees and Quadrant II when measured in radians, so that the degree measure in QI will equal radian measure of the reference angle in QII. Thus, we only need to solve: $x=\pi-x\cdot\frac{\pi}{180}$, with the $x\cdot\frac{\pi}{180}$ term representing the radian measure of angle $x°$. Solving gives $x=\frac{180\pi}{180+\pi}$. Answer: $\frac{180\pi}{180+\pi}$
20.	Since the second digit is twice the first, it must be even and therefore can only be 2 or 4. If it is a 2, then the first digit must be 1. If 1 is in the fifth position, then the first digit must be 2. The other statements need not be true. Answer: A