Fall 1999

1) Given the following 5 statements: 1) All women are good drivers. 2) Some women are good drivers. 3) No men are good drivers. 4) At least one man is a bad driver. 5) All men are good drivers. The statement that negates 5 is:

A) 1 B) 2 C) 3 D) 4

2) How many three digit numbers can be written using no 0's and at least one 7?

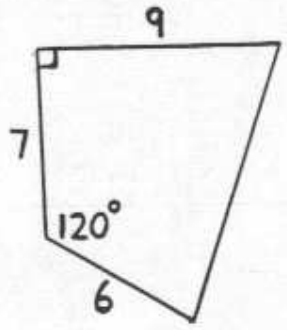
3) For the set {40, 90, 60, 50, x}, the mean, mode, and median are all equal. Find x.

4) If 3 blots equal 4 bleets and 5 bleets equal 6 blits, what is the ratio of one blit to one blot?

5) Bill can run around an oval track in 40 seconds. Sue, running in the opposite direction, meets Bill every 15 seconds. What is Sue's time around the track in seconds?

A) $12\frac{1}{2}$ B) 24 C) $27\frac{1}{2}$ D) 55

6) Find the length of the longest line segment that can be drawn in (or on) the figure shown below.



7) Suppose f, g, and h are functions defined for all real numbers such that $f(x) = g(x - 1)$ and $g(x) = h(2 - x)$. Which of the following must be true?

A) $h(x) = f(4 - x)$ B) $h(x) = f(1 - 2x)$
C) $h(x) = f(3 - x)$ D) $h(x) = f(x - 4)$

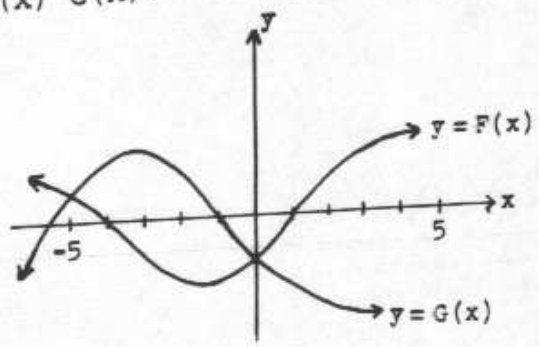
8) The largest prime factor of $41! + 42! + 43!$ is:

A) 43 B) 37 C) 47 D) 41

9) A bag contains 5 blue marbles, 4 white marbles, and 3 red marbles, all identical except for color. If 3 marbles are randomly selected without replacement, what is the probability that the marbles will be the same color?

10) Five robots produce 5 automotive parts in 5 minutes. How many packages, each containing one part, could be produced by 10 robots in 10 hours?

11) Find all values of x where $[F(x)]^2 = F(x)\ G(x)$.



12) Evaluate $3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - 2}}}} \ldots$

A) 2 B) 0 C) $\frac{2}{3}$ D) $\frac{7}{3}$

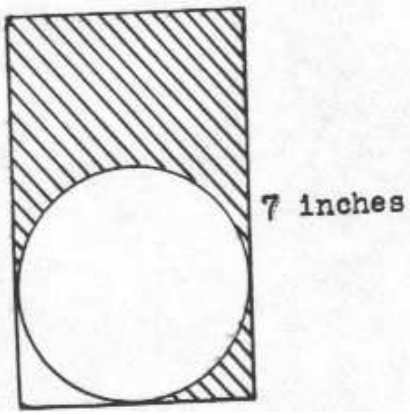
13) In the expansion of $(a + b)^n$ there are $n + 1$ dissimilar terms. The number of dissimilar terms in the expansion of $(a + b + c)^{10}$ is:

A) 11 B) 33 C) 66 D) 132

14) Vertical line segments are drawn at each of the positive integers, connecting the graphs of $y = \left(\frac{3}{4}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$. The sum of the lengths of these line segments is:

A) $\frac{1}{2}$ B) 1 C) 2 D) 12

15) Find the area of the shaded region shown. The circle just fits in the rectangle and is tangent on 3 sides.



16) If you add the 0^{th}, 1^{st}, 3^{rd}, and 4^{th} powers of a positive prime number, then the result cannot be:

A) An odd number
B) An even number
C) Less than 50
D) A perfect square

17) Exactly one number is in the set $A \cap B \cap C \cap D$ where sets A, B, C, and D are as defined. What is the number?

A = {9, 18, 27, 36, 45, 54, 63, . . .}
B = {44, 144, 244, 344, 444, 544, 644, 744, 844, 944, 1044, 1144, . . .}
C = {3, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 300, 301, 302, . . .}
D = {1000, 1001, 1002, 1003, 1004, . . ., 9999}

18) Which of the following numbers is the largest?

A) 125 (126) (127) (128) . . . (374)
B) 250^{250}
C) $2^{(2^{11})}$
D) 5^{875}

19) An exponential expression has a base of 4 and a positive exponent. If the base is doubled, by what percent would the exponent be decreased in order to leave the value unchanged?

20) The fraction $\frac{a^2 + b^2 - c^2 + 2ab}{a^2 + c^2 - b^2 + 2ac}$ is (with suitable restrictions on a, b, and c):

A) Irreducible
B) Reducible to -1
C) Reducible to $\frac{a - b + c}{a + c - b}$
D) Reducible to $\frac{a + b - c}{a + c - b}$

SOLUTIONS
FALL 99

1) THE NEGATION OF "ALL... ARE...." IS "AT LEAST ONE... IS NOT..."

ANS D

2) SINCE THERE ARE NO 0's, THERE ARE 9 DIGITS LEFT.

EXACTLY 1 SEVEN $3(1\cdot 8\cdot 8) = 192$
EXACTLY 2 SEVENS $3(1\cdot 1\cdot 8) = 24$
THREE SEVENS $1\cdot 1\cdot 1 = 1$
217

ANS 217

3) THE SCORES IN ORDER ARE $\{40, 50, x, 60, 90\}$
FOR x TO BE THE MEDIAN. $40+50+60+90 = 240$
IN ORDER FOR THERE TO BE A MODE ONE NUMBER MUST REPEAT. IF $x=60$, THEN 60 IS MODE AND MEDIAN. FURTHER, $\frac{240+60}{5} = \frac{300}{5} = 60$

ANS 60

4) BY DIVIDING 1 BLET $= \frac{4}{3}$ BLEET AND 1 BLIT $= \frac{5}{6}$ BLEET.

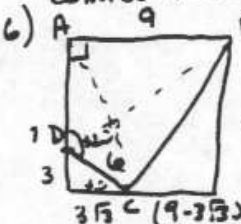
THUS, $\frac{\text{BLIT}}{\text{BLOT}} = \frac{\frac{5}{6}}{\frac{4}{3}} = \frac{5}{6}\times\frac{3}{4} = \frac{5}{8}$ ANS B

5) LET x = SUE'S TIME IN SEC. BILL COMPLETES $\frac{15}{40}$ OF A LAP, WHILE SUE COMPLETES $\frac{15}{x}$ OF A LAP. SINCE THEY COMPLETE A LAP IN 15 SEC.

$\frac{15}{40} + \frac{15}{x} = 1 \Rightarrow \frac{15}{x} = \frac{5}{8} \Rightarrow x = 24$ SEC.

ANS. B

6) COMPLETING THE RECTANGLE WE HAVE



THE CHOICES WOULD BE

$|BD| = \sqrt{7^2+9^2} = \sqrt{130}$
$|AC| = \sqrt{10^2+(3\sqrt{3})^2} = \sqrt{127}$
$|BC| = \sqrt{10^2+(9-\sqrt{3})^2} = \sqrt{209-54\sqrt{3}}$

ANS $\sqrt{130}$

7) $g(2-x) = h(2-(2-x)) = h(x)$
$f(x+1) = g((x+1)-1) = g(x)$
THUS, $h(x) = g(2-x) = f(2-x+1) = f(3-x)$

ANS. C

8) $41! + 42! + 43! = 41!(1 + 42 + 43\cdot 42)$
$= 41!(43 + 43\cdot 42) = 41!(43(1+42))$
$= 41!(43)(43) = 41!(43)^2$

9) THERE ARE $\binom{12}{3} = \frac{12!}{3!\,9!} = 220$ WAYS TO PICK 3 MARBLES FROM 12.
TO PICK 3 BLUE $\binom{5}{3} = 10$ WAYS
TO PICK 3 WHITE $\binom{4}{3} = 4$ "
TO PICK 3 RED $\binom{3}{3} = 1$
PROB. (SAME COLOR) $= \frac{10+4+1}{220} = \frac{15}{220} = \frac{3}{44}$

ANS. $\frac{3}{44}$

10) 5 ZOOTS PRODUCE $\frac{5 \text{ PARTS}}{5 \text{ MIN}} = \frac{1 \text{ PART}}{1 \text{ MIN}}$

1 ROBIT $= \frac{1}{5}\frac{\text{PART}}{\text{MIN}}$
10 ROBIT $= 2\frac{\text{PART}}{\text{MIN}}$
$2\frac{\text{PART}}{\text{MIN}}(10\cdot 60 \text{ MIN}) = 1200$ PARTS

ANS. 1200 PARTS

11) $f(-6)\,g(-5) = f(-5)^2$ EQUALITY
$f(-4) = 0 \Rightarrow f(-4)^2 = f(-4)g(-4) = 0$
$f(0) = g(0) \Rightarrow [f(0)]^2 = f(0)g(0)$
$f(1) = 0 \Rightarrow [f(1)]^2 = f(1)g(1) = 0$

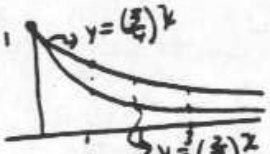
ANS. $x = -5, -4, 0, 1$

12) LET $x = 3 - \cfrac{2}{3 - \cfrac{2}{3-2+\cdots}}$
THUS $x = 3 - \frac{2}{x}$
$x^2 = 3x - 2$
$x^2 - 3x + 2 = 0$
$(x-2)(x-1) = 0$
$x = 2 \mid x = 1$

ANS. $x = 2$ A

13) ON NEXT PAGE

14)



$(\frac{3}{4} - \frac{2}{3}) + \left[(\frac{3}{4})^2 - (\frac{2}{3})^2\right] + \left[(\frac{3}{4})^3 - (\frac{2}{3})^3\right] + \cdots$
$= \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3 + \cdots - (\frac{2}{3} + (\frac{2}{3})^2 + (\frac{2}{3})^3 + \cdots)$
IF $S = \frac{3}{4} + (\frac{3}{4})^2 + (\frac{3}{4})^3 + \cdots$
THEN $\frac{3}{4}S = (\frac{3}{4})^2 + (\frac{3}{4})^3 + \cdots$
$\Rightarrow \frac{1}{4}S = \frac{3}{4} \Rightarrow S = 3$

SIMILARLY $\frac{2}{3} + (\frac{2}{3})^2 + \cdots = 2$

THUS, $3 - 2 = 1$ ANS. B

15) THE AREA OF CIRCLE IS $A_O = \pi(2)^2 = 4\pi$

CONSIDER THE SQUARE REGION  4"

4"

$A_{SQ} = 4^2 = 16$

$A_{SQ} - A_O = 16 - 4\pi$ GIVES THE AREA OF 4 $\triangle$ PIECES

AREA OF 1 $\triangle$ $= \frac{16-4\pi}{4} = 4 - \pi$

$A_O + A_{\triangle} = 4\pi + (4-\pi) = (4+3\pi)$

$A_{RECT} = 4(7) = 28$

$A_{SHADED} = 28 - (4+3\pi) = (24-3\pi)\text{ in}^2$

16) $1 + p + p^3 + p^4$ WHERE p IS PRIME.

IF $p=2$, RESULT IS ODD

IF $p=3$, RESULT IS EVEN.

IF $p \geq 2$, WE HAVE $1+2+8+16 = 27 < 50$

$1+p+p^3+p^4 = 1+p^2+p+p^4 = (1+p^3) + p(1+p^3)$

$= (1+p^3)(1+p)$

$= (1+p)(1-p+p^2)(1+p)$

$= (1+p)^2(1-p+p^2)$

$1-p+p^2$ CAN NOT BE A PERFECT SQUARE

ANS. D

17) A: MULTIPLES OF 9 B: DIGITS ENDING IN 44

C: NUMBERS WITH 3 AS A FIRST DIGIT

D: NUMBERS BETWEEN 1000 & 9999.

THUS, 3_44. IN ORDER TO BE A MULTIPLE OF 9, THE SUM OF THE DIGITS MUST BE DIVISIBLE BY 9. $3+x+4+4 = 11+x \Rightarrow x=7$

ANS. 3744

18) $2^{2^{11}} = 2^{2048} = (2^8)^{256} = 256^{256}$ CLEARLY C IS BIGGER THAN B. IT IS BIGGER THAN A WHICH HAS 250 NUMBERS AND $(125)(374) < 250^2$

FURTHER, $5^{875} < 5^{876} = (5^3)^{292} = 125^{292} < 128^{292}$

$\Rightarrow 5^{875} < 128^{292} = (2^7)^{292} = 2^{2044} < 2^{2048}$

ANS. C

19) LET a AND b BE THE EXPONENTS SO THAT $4^a = 8^b$

$(2^2)^a = (2^3)^b$

$2^{2a} = 2^{3b} \Rightarrow 3b = 2a$

$4^a = 8^{\frac{2}{3}a}$

a SHOULD BE DECREASED BY

$33\frac{1}{3}\%$ ANS.

20) $\frac{a^2+b^2-c^2+2ab}{a^2+c^2-b^2+2ac}$

$= \frac{a^2+2ab+b^2-c^2}{a^2+2ac+c^2-b^2}$

$= \frac{(a+b)^2-c^2}{(a+c)^2-b^2}$ DIFFERENCE OF 2 SQ. TERMS

$= \frac{[(a+b)+c][(a+b)-c]}{[(a+c)+b][(a+c)-b]}$

$= \frac{(a+b+c)[a+b-c]}{(a+b+c)[a+c-b]}$

ANS: $\frac{[a+b-c]}{a+c-b}$ ANS. D

13) $(a+b+c)^{10} = [(a+b)+c]^{10}$

$= (a+b)^{10} + (a+b)^9 c + \_(a+b)^{8}c^2$

$+ \ldots + c^{10}$

THE NUMBER OF TERMS IN EACH IS: $11+10+9+8+7+6+5+4+3+2+1$

$= 66$

ANS. C