Math Contest Spring 2008

1. If the graph of $f(x) = e^x$ is shifted to the right 3 units and up four units, then the equation of the shifted graph would be $g(x) =$ ___________?

2. For how many prime numbers, p, can the difference of the cube of p and p itself be expressed as the product of three consecutive integers?
   A. 0 B. 1 C. 13 D. All primes

3. Evaluate the expression:

$$\frac{87!}{84!(3!)} - \frac{86!}{83!(3!)}$$

4. The radius of one circle is three times that of a second circle. If they are drawn in the same plane so that the distance between their centers is 1 1/3 times the radius of the larger circle, in how many places do the two circles intersect?

5. Find the value of $-x^{-1/x}$ if $x = (-2)^{-2}$.

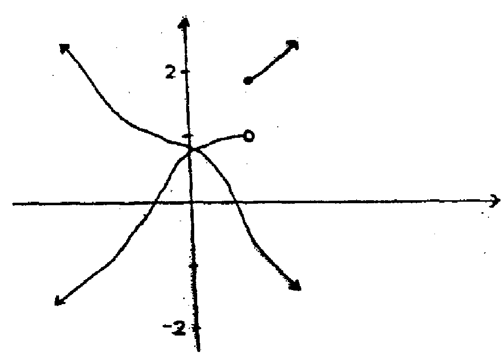
6. For light, intensity varies inversely as the square of the distance. If the distance from a light source is 2.5 units, then what percentage of the intensity remains compared to one unit away?

7. A high school sports awards banquet for boys at Sorran High is an annual event. From past experience the caterer knows that 100% of the boys less than 25 years old will get second helpings from the buffet. 40% of those men between 25 and 55 years old inclusive will get a second helping. Only 5% of those men over 55 will get a second helping. At this years dinner there are 50 boys under 25, 40 men between 25 and 55 inclusive, and 20 men over 55 years old. If a male gets a second helping, then what is the probability that he is over 55?

8. How many numbers between $\frac{1}{8}$ and $\frac{1}{2}$ inclusive can be displayed exactly on an 8 digit display calculator? Assume there is not a leading zero and the decimal point does not take one of the eight spaces.

9. f is increasing on $(-\infty, \infty)$ and g is decreasing on $(-\infty, \infty)$. Their graphs are shown. Which of the following is true about the maximum value of the product, $(fg)(x)$?

A. It's in $(-\infty, 0)$. B. It's in $[0, 1)$. C. It's in $[1, 3)$.
D. It's in $[3, \infty)$



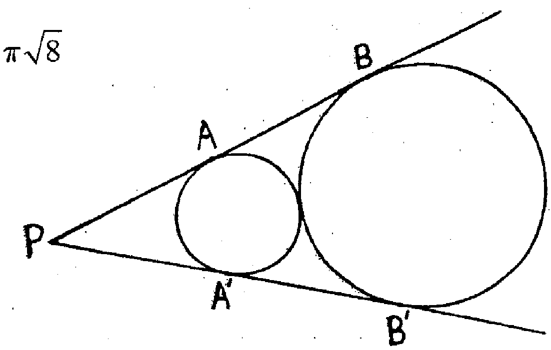
10. Tina was trying to evaluate $2^x + 4^x$ for some particular real number x. She accidentally evaluated $(2^x)(4^x)$ instead. However, she got the correct value for $2^x + 4^x$. Find the value of x, rounded to 7 decimal places or the exact value.

11. One hundred students at Five Score High School participated in a math contest and their mean score was 100. The number of non-seniors taking the contest was 50% more than the number of seniors taking the contest, and the mean score of the seniors was 50% more than that of the non-seniors. What was the mean score of the seniors?

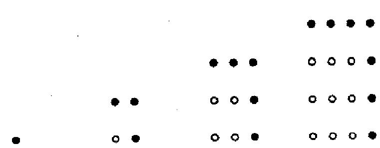
A. 100 B. 112.5 C. 120 D. 125

12. Two circles are externally tangent. Lines PAB and PA'B' are common tangents with A and A' on the smaller circle and B and B' on the larger circle. If PA = AB = 4, then the area of the smaller circle is

A. 1.44π B. 2π C. 2.56π D. $\pi\sqrt{8}$



13. Picker Plucker Orchards always plant new trees in their apple orchard so that the arrangement of the trees forms a square as shown below. If the orchard adds 33 trees to one field, then how many trees are in that field after the trees are added?



1 4 9 16 Note: The • denotes a new tree.

14. Find the smallest value of n for which k is a positive integer when:

$3^n - 1 = 11k$.

15. For θ any real number, which of the following can never be greater than 1?

A. $\dfrac{\theta^2 - 1}{\theta^2 - 2}$ B. $\sin\theta - \cos\theta$ C. $\dfrac{\sin\theta}{\sec\theta}$ D. $\sin\theta + \cos\theta$

16. A Pythagorean Triple is a triple (a, b, c), where a, b, and c are positive integers that satisfy $a^2 + b^2 = c^2$. How many such distinct Pythagorean Triples exist that have 15 as one of the three values?

A. 1 B. 4 C. 5 D. More than 5

17. Let $f(x) = x^2 - 2x$. What is the sum of all the solutions to the equation, $f(x - f(x)) = f(f(x))$?

18. There are n dimes in a jar. Consider the following 5 statements about how many dimes there are in the jar.

a. The number of dimes is a prime number.
b. There are at least 7 dimes.
c. There are less than 17 dimes.
d. The number of dimes is a multiple of 3.
e. There are an odd number of dimes.

What is the sum of all possible values of n such that exactly 4 of the above statements is true?

19. The six digit number 1k31k4 has a factor of 12 but it is not divisible by 9. Find the digit k.

20. At how many points do the graphs of $x^2 + y^2 = 2^{200000}$ and $x^3 - y^3 = 4^{150001}$ intersect?

A. 0 B. Exactly 1 C. Exactly 2 D. More than 2

Math Contest Solutions
Spring 2008

1. In order to shift the graph to the right 3 units, we need to subtract 3 from x. In order to move the graph up 4 units we need to add 4 to each y coordinate. Hence, the new function g(x) will be $g(x) = e^{x-3} + 4$. **Ans. $g(x) = e^{x-3} + 4$**

2. Let p be a prime number. Then p cubed minus p means $p^3 - p = p(p^2 - 1) = p(p + 1)(p - 1) = (p - 1)(p)(p + 1)$. The last expression is the product of three consecutive integers for any prime, p. **Ans. D**

3. $\frac{87!}{84!(3!)} - \frac{86!}{83!(3!)} = \frac{87(86)(85)(84!)}{84!(3)(2)(1)} - \frac{86(85)(84)(83!)}{83!(3)(2)(1)} =$
$\frac{87(86)(85)}{6} - \frac{86(85)(84)}{6} = \frac{86(85)}{6}(87 - 84) = \frac{86(85)(3)}{6} = 3655$. **Ans. 3655**

4. We can assume without loss of generality that the radius of the smaller circle is 1 unit and the larger radius is 3 units. The distance that is $1\frac{1}{3}$ times the radius 3 is 4 units. Hence, the center of the smaller circle must lie on the circumference of a circle of radius 4 that has the same center as the circle of radius 3. Since the radius of the smaller circle is 1, the smaller circle must be tangent to the larger one. Hence, they intersect in exactly one point. **Ans. 1 point**

5. If $x = (-2)^{-2} = 1/(-2)^2 = \frac{1}{4}$, then $\frac{-1}{x} = \frac{-1}{\frac{1}{4}} = -4$. Thus, $-x^{-1/x} = -\frac{1}{4}^{-4} = -4^4$ -256.

Ans. -256

6. If I = light intensity and d = distance, we have $I = \frac{k}{d(d)}$ for some constant, k. For a distance of 1 unit, we have $I = \frac{k}{1(1)} = k$. For a distance of 2.5 units we have $I = \frac{k}{2.5(2.5)} = \frac{k}{6.25} = \frac{I}{6.25}$. Hence, the intensity at one unit has been decreased by the fraction $\frac{1}{6.25} = .16 = 16\%$. **Ans. 16%**

7. We need to know the number of males that will get a second helping. 100% of those under 25 will have 50 getting a second helping. Of the men between 25 and

55 we have 40% of 40 men or 16 getting seconds. Lastly, 5% of those over 55 we have 5% of 20 which is 1 will have seconds. Thus, there are 50 + 16 + 1 = 67 getting seconds.

Thus, the chance that the person is over 55 is $\frac{1}{67}$. **Ans.** $\frac{1}{67}$

8. $\frac{1}{8}$ = .125 and $\frac{1}{2}$ = .5. If the first two digits after the decimal point are 1 and 2 in that order, then the third digit must be five or higher (5 choices). The five remaining digits could be any of the 10 digits. The number of possibilities is 1(1)(5)(10)(10)(10)(10)(10) = 500,000. Another case is to have a 1 in the first place, the second three or more (7 choices), and the remaining digits could be any of the 10. The number of possibilities for this is 1(7)(10)(10)(10)(10)(10)(10) = 7,000,000. Next we could have the first digit be 2, 3, or 4 (3 choices) and the remaining 7 digits could be any of the 10. The number of possibilities is 3(10)(10)(10)(10)(10)(10)(10) = 30,000,000. Lastly, we could have .5 itself, 1 case. The total is the sum of these or 37,500,001. Another solution is to ignore the decimal point and consider all the numbers between 12,500,000 and 50,000,000. The difference of these two numbers is 37,500,000. If we add one to this for the first number, we get 37,500,001. **Ans. 37,500,001**
9. Since f is the increasing function, it has an x-intercept for some negative number. To the left of this intercept f has negative function values while g has positive values. Thus, the product of values will be negative to the left of the f x-intercept. The decreasing function g, has an x-intercept for some positive value of x. To the right of this intercept, the function, g, is positive while the function, f, is negative. Again, the product function will be negative to the right of this intercept. Between the two x-intercepts, both functions are positive so their product will be positive. A glance at the function values shows that their product will be less than one. Hence, the answer is choice B.
Ans. B
10. Let $2^x + 4^x = 2^x(4^x)$. Then, $4 = 2^2$ so $4^x = (2^2)^x = 2^{2x}$. On subbing we have $2^x + 2^{2x} = 2^x\,2^{2x}$ or $2^x + 2^{2x} = 2^{3x}$. Rewriting we have $2^{3x} - 2^{2x} - 2^x = 0$. On factoring we have $2^x(2^{2x} - 2^x - 1) = 0$. 2^x 0. So $2^{2x} - 2^x - 1 = 0$.If we let $y = 2^x$, then the equation becomes $y^2 - y - 1 = 0$. If we use the Quadratic Formula, we find that y = (1 $\sqrt{5}$)/2. Since $y = 2^x$, we know y is positive. Thus, y = (1 + $\sqrt{5}$)/2. Then 2^x = (1 + $\sqrt{5}$)/2 . If we take the ln of both sides we have xln2 = ln (1 + $\sqrt{5}$)/2. So that x = (ln (1 + $\sqrt{5}$)/2)/ln2 = .6942419 to 7 places. **Ans. .6942419**
11. Since the total number of students was 100 and their mean score was also 100,

the total of all scores was 100(100) = 10,000. If we let s = the number of seniors,

then the number of non-seniors is 50% more or 1.5s. The total number of students is 100 so we have s + 1.5s = 100. 2.5 s = 100, s = 40 and 1.5s = 60. Thus, there were 40 seniors and 60 non-seniors. If we let m = the mean score of

the non-seniors, then since the seniors mean score was 50% higher, the mean senior score was 1.5y. The 60 non-seniors had a total score of 60y while the 40

seniors had a score of 40(1.5y) = 60y also. Thus, 60y + 60y = 10,000. then, y =

10000/120 = 250/3. The mean score of the seniors is then $1.5(\frac{250}{3}) = \frac{3}{2}(\frac{250}{3}) =$

125. **Ans. D**

12. If we draw a line from P through the centers of the two circles, we can form two right triangles by drawing radii from the centers to the point of tangency at A and A'. The two triangles are similar so the ratio of corresponding sides is the same. If we let r = radius of the smaller circle and R = radius of the larger circle, we have $\frac{4}{4+4} = \frac{r}{R}$. This gives us R = 2r. The distance between the centers of the circles is r + R. By similar triangles this is the distance from P to the center of the smaller circle. This line segment is also the hypotenuse of the triangle formed from P to A to the center of the smaller circle. By the Pythagorean Theorem, we have $4^2 + r^2 = (3r)^2$. The solution is $r = \sqrt{2}$. Hence, the area of the smaller circle is $\pi r^2 = \pi(\sqrt{2})^2 = 2\pi$. **Ans. 2π un.^{2}**
13. If we follow the pattern, for the second square there were 2 + 1 trees added, for the third square there were 3 + 2 trees added, for the fourth square there were 4 + 3 trees added and so forth. In each case we had the number of the square added to one less than the number of the square, or n + n – 1. Thus, n + n – 1= 33. So that n = 17. The total number of trees will be $17^2 = 289$. **Ans. 289**
14. Since k must be positive we can eliminate n = 0. The first several powers of three are 3, 9, 27, 81, 243, and 729 for which $3^n - 1$ is 2, 8, 26, 80, 242, and 728. The first multiple of 11 is 242 which is 11(22). Thus, k = 22 and n = 5. **Ans. 5**
15. For θ = 2, $(2^2 - 1)/(2^2 - 2) = 3/2 > 1$. For θ = 3π/4, sin3π/4 - cox3π/4 = $\sqrt{2}/2 - (-\sqrt{2}/2) = \sqrt{2} > 1$. For θ = π/4, we have sin θ + cos θ = $\sqrt{2}/2 + (\sqrt{2}/2) = \sqrt{2} > 1$. The only possible answer is C. **Ans. C**

16. If $a^2 + b^2 = c^2$ is true for some triple of values a, b, and c, then so is ka, kb, and kc for any positive k. Since 3, 4, and 5 form a Pythagorean Triple, then so do 15,

20, and 25 as well as 9, 12, and 15. The last is the only triple for which 15 plays the role of c. If we let a = 15, then we can write $15^2 + b^2 = c^2$. Then, $c^2 - b^2 = 225$. We can factor this so that we have $3^2(5^2) = (c + b)(c - b)$. If we let c + b = 25 and c – b = 9, we find that c = 17 and b = 8. Thus, we have the triple 8, 15 and 17. If let c + b = 45 and c – b = 5, we find that c = 25 and b = 20 which repeats a previous triple. If we let c + b = 75 and c – b = 3, then we find that c = 39 and b = 36. Thus, we have 15, 36, and 39. Lastly, if we let c + b = 225 and c – b = 1, we find that c = 113 and b = 112. Thus, we have the triple 15, 112, and 113. Since the only factors of 15 are 3 and 5, we have all the cases. Thus, there are 5 Pythagorean Triples which include 15. **Ans. 5**

17. Since $f(x) = x^2 - 2x$, we have $f(x - f(x)) = f(x - (x^2 - 2x)) = f(3x - x^2) = (3x - x^2)^2 - 2(3x - x^2) = 9x^2 - 6x^3 + x^4 - 6x + 2x^2 =$. Also, we have $f(f(x)) = f(x^2 - 2x) = (x^2 - 2x)^2 - 2(x^2 - 2x) = x^4 - 4x^3 + 4x^2 - 2x^2 + 4x = x^4 - 4x^3 + 2x^2 + 4x$. Equating these two expressions we have $11x^2 - 6x^3 + x^4 - 6x = x^4 - 4x^3 + 2x^2 + 4x$. Getting one side to be 0, we have $2x^3 - 9x^2 + 10x = 0$. By factoring, we get $x(2x - 5)(x - 2) = 0$. The solutions to this are 0, 2 and $\frac{5}{2}$. The sum of the solutions is $\frac{9}{2}$. **Ans.** $\frac{9}{2}$

18. There are no values greater than 17 because any value greater than 17 would have to be prime and a multiple of 3 which can not happen. 3 is a value less than 7 that is prime, odd, and a multiple of 3. The other values must be in the interval, $7 \le n < 17$. The values of n in this interval that make 4 out 5 statements true are 7, 9, 11, 13, and 15. The sum of all possible values is 58. **Ans. 58**

19. Since the number has a four in the ones place it is divisible by 2. In order for it to be divisible by 4, k would have to be 0, 2, 4, 6, or 8. For the number to be divisible by 3, the sum of the digits has to be a multiple of 3. So, 1+k+3+1+k +4 = 2k + 9 = 3n for some integer n. Since 9 is a multiple of 3, 2k has to be a multiple of 3. Thus, k = 0, 3, 6, or 9. For k = 0 or 9, the sum of the digits would be divisible by 9 which is not allowed. For k = 3, the last two digits would be 34 which would make the number not divisible by 4. Hence, k = 6. **Ans. k = 6**

20. The graph of $x^2 + y^2 = 2^{200000}$ is a circle centered at the origin of radius $r = 2^{100000}$. Upon solving the second equation for y we have $y = (x^3 - 4^{150001})^{1/3}$. The graph of this expression has its intercept at $x = (4^{150001})^{1/3} = (2^{300002})^{1/3} > 2^{100000}$.

For $x < (4^{150001})^{1/3}$, y will be less than 0. The y-intercept will be at $y = (-4^{150001})^{1/3} = (-2^{300002})^{1/3} < -2^{100000}$. Hence, both the x and the y-intercepts are outside the circle. Also, for the circle, $-2^{100000} \leq x \leq 2^{100000}$ and $-2^{100000} \leq y \leq 2^{100000}$. If we cube each of these inequalities, we have -2^{300000} x^3 2^{300000} and $-2^{300000} \leq y^3 \leq 2^{300000}$. In particular, we have x^3 2^{300000} and $-y^3$ 2^{300000}. Adding these last two we get $x^3 - y^3$ $2(2^{300000}) = 2^{300001} < 2^{300002}$. Thus, the values of x and y on the circle never get as large as the values of x and y on the second curve. Thus, there are no points of intersection. **Ans. A**